SEQUENCE SPACES AND MEASURES OF NONCOMPACTNESS

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By

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Under the supervision of
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Dedicated to My Parents

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Certificate

Certified that ***Mr. Vakeel Ahmad Khan*** has carried out research work on ***"SEQUENCE SPACES AND MEASURES OF NONCOMPACTNESS"*** and is suitable for submission for the award of the degree of Master of Philosophy in Mathematics.

It is further certified that this work has not been submitted in any university or institution for the award of any other degree or diploma.

Dr. Mursaleen
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PREFACE

The present dissertation entitled "**SEQUENCE SPACES AND MEASURES OF NONCOMPACTNESS**" has been compiled for submission to Aligarh Muslim University in partial fulfilment of the requirements of the degree of Master of Philosophy .

The dissertation consists of five chapters. In chapter I, we recall notations, concept of linear metric and paranormed spaces, concept of Shauder basis and some topological concepts.

In chapter II, we discuss the definition and results on FK spaces, and matrix transformations in to l_∞, c, and c_0.

In chapter III, we study the $\alpha-$, $\beta-$, and continuous duals of the classical sequence spaces.

In chapter IV, we study the Maddox sequence spaces and their duals.

Finally the last chapter deals with the study of the Kuratowski measure of noncompactness, the Hausdorff measure of noncompactness, the inner Hausdorff measure of noncompactness, the Istrăţescu's measure of noncompactness and measure of noncompactness of an operator.

Towards the end of the dissertation, we have given a fairly exhaustive bibliography of the books and publications to which reference have been made throughout the dissertation.

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CONTENTS

CHAPTER I

PRELIMINARIES

1.1. Introduction

The primary aim of this chapter is to give some notations and basic definitions on basic sequence spaces.

We also recall some concepts related to our study such as linear metric spaces, paranormed spaces, Schauder basis etc. some topological concepts are also given which form the background to study the concept of measures of noncompactness which is the central theme of our present study.

Most of the concepts used here are taken from Maddox [26], [27], Wilansky [37]. Malkowsky [29], Marti [30], Lindenstrauss [21], and Kuratowski [17].

1.2. Notations

Throughout the present work we shall use the following notations which are conventional (cf.Cook[3], Hardy [9], Maddox [27]).

$\mathbb{N}$:= The set of all natural numbers

$\mathbb{R}$:= The set of all real numbers

$\mathbb{C}$:= The set of all complex numbers

Q:= The set of all rational numbers

$\lim_k$: means $\lim_{k\to\infty}$

$\inf_k$: means $\inf_{k\geq 1}$, unless otherwise stated

$\sup_k$: means $\sup_{k\geq 1}$, unless otherwise stated

$\sum_k$: means summation over k = 1 to $k=\infty$, unless otherwise stated

X := (x_k) or $\{x_k\}$, the sequence whose *k-th* term is x_k

e_k:=(0,0,……0,1,0,…..), the sequence whose *k-th* component is 1 and others zeros, for all $k \in \mathbb{N}$

e:= (1,1,1,……)

ω:= $\{x=(x_k)$: $x_k \in \mathbb{R}$ or $\mathbb{C}\}$, the space of all sequences, real or complex

l_∞:= $\{x \in \omega : \sup_k |x_k| < \infty\}$, the space of all bounded sequences

c:= $\{x \in \omega : \lim_k x_k = l$, for some $l \in \mathbb{C}\}$ the space of all convergent sequences

c_0:= $\{x \in \omega : \lim_k x_k = 0\}$, the space of all null sequences

l_∞, c, and c_0 are Banach spaces with the norm $\|x\|_\infty = \sup_k |x_k|$

φ:= The space of all finitely non-zero sequences

We have $\varphi \subset c_0 \subset c \subset l_\infty$ the inclusion being strict.

$\zeta := \{x = (x_k) : |x_n - x_m| \to 0,$ as $m, n \to \infty\}$, the set of cauchy sequence.

Then we have $c \subset \zeta \subset l_\infty$

$cs :=$ $\{x \in \omega : \sum_k x_k$ converges$\}$, the space of convergent series

$l_1 :=$ $\{a : (a_k) : \sum_k |a_k| < \infty\}$, the space of absolutely convergent series

bs := $\{x \in \omega : \left(\sum_{k=0}^{n} x_k\right)_{n=0}^{\infty} \in l_\infty\}$, the space of bounded series

l_p := $\{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$, for $1 \leq p < \infty$

Then we have $l_1 \subset cs \subset c_0 \subset c \subset \zeta \subset l_\infty$, the inclusion being strict.

$bv := \{x : (x_k) : \sum_k |x_k - x_{k+1}| < \infty, x_0 = 0\}$, the set of sequences of bounded variation

L(X, Y) := The set of all linear operators on X into Y

B(X, Y) := The set of all bounded (i.e. continuous) linear operators on X into Y

$X^\dagger$:= The set of all linear functionals on X and it is called the algebraic dual space of X

X^* := The set of all bounded linear functionals on X and it is called the dual space of X

$X^* \subset X^\dagger$, strictly where X=l_1 with the norm of l_∞

$X^* = X^\dagger$ for finite dimensional spaces.

Let $p=(p_k)$ be a sequence of positive real numbers (not necessarily bounded in general). The following sequence spaces were defined and studied by Maddox [26], Lescarides and Maddox [20], Simons [35]

$l_\infty(p) := \{x : \sup |x_k|^{p_k} < \infty\}$

$c_0(p) := \{x : |x_k|^{p_k} \longrightarrow 0 \text{ as } k \to \infty\}$

$c(p) := \{x : |x_k - l|^{p_k} \longrightarrow 0 \text{ for some } l \in \mathbb{C}\}$

$l(p) := \{x : \sum |x_k|^{pk} < \infty\}$

$M(p) := \{a : \sum |a_k|^{q_k} N^{-q_k} < \infty$ for some integer N>1$\}$ with

$p_k^{-1} + q_k^{-1} = 1$

If $0 < p_k \leq 1$ then $l^\dagger(\text{p}) = l_\infty(p)$

If $p_k > 1$ then $l^\dagger(\text{p}) = \text{M}(p)$

If p_k= p for all k than $l_\infty(p) = l_\infty$

$c_0(p) = c_0$, $c(p) = c$ and $l(p) = lp$

It was shown that the above sequence spaces are linear spaces under coordinate-wise addition and scalar multiplication if and only if $p \in l_\infty$ Whenever $p \in l_\infty$ we shall write

$H = \sup_k p_k$ and M=max(1,H)

1.3. Linear Metric and paranormed spaces

Let X be a linear space and d a metric on X. Then (X,d) or X in short, is said to be a linear metric space, if the algebraic operations on X are continuous functions.

A complete linear metric space is said to be a Frechet space (c.f.[36], Definition 5.3.2, p.78). (Unfortunately this terminology is not universally agreed on. Some authors call a complete linear metric space an F-space and a locally convex F-space a Frechet space (See e.g. [34, p.8], [16, p.208], which Wilansky calls an F-space. The continuity of the algebraic operations of a linear metric space (X,d) means the following if (x_n) and (y_n) are two sequences in X and (λ_n) is a sequence of scalars with $x_n \to x$, $y_n \to y$ and $\lambda_n \to \lambda$ $(n\to\infty)$ then $x_n+y_n \to x+y$ $(n\to\infty)$ and $\lambda_n x_n \to \lambda x (n\to\infty)$, this means that $d(x_n,x)$, $d(y_n,y)\to 0$ and $\lambda_n \to \lambda (n\to\infty)$ together imply $d(x_n+y_n, x+y)\to 0$ and $d(\lambda_n x_n, \lambda x)\to 0$ $(n\to\infty)$.

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. The paranorm of a vector x may be thought of as the distance x to the origin 0.

Definition 1.3.2. Let X be a linear space. A function p:X→ℝ is called *Paranorm*, if

(P.1) $p(0) = 0$

(P.2) $p(x) \geq 0$ for all $x \in X$

(P.3) $p(-x) = p(x)$ for all $x \in X$

(P.4) $p(x+y) < p(x)+p(y)$ for all $x, y \in X$ (triangle inequality)

(P.5) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda (n \to \infty)$ is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$ then $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ (continuity of multiplication by scalars).

If p is a paranorm on X then (X,p), or X in short is called a *paranormed space.* A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total.* For any two paranorms p and q, p is called stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$ $(n \to \infty)$, then also $q(x_n) \to 0$ $(n \to \infty)$. If p is a stronger than q, then q is said to be weaker than p. If p is stronger than q, and q is stronger than p, then p and q are called *equivalent*. If p is stronger than q, but p and q are not equivalent, then p is said to be *strictly stronger* than q, q is called *strictly weaker* than p.

It is easy to see that every totally paranormed space is a linear metric space. The converse is also true. The metric of any linear metric space is given by some total paranorm (cf.[36], Theorem 10.4.2, p.183]). A sequence of paranorms may be used to define a paranorm.

Theorem. 1.3.3. Let $(p_k)_{k=1}^{\infty}$ be a sequence of paranorms on a linear space X. we define the so called Frechet combination of (p_k) by

(1.3.3.1) $$p(x)=\sum_{n=0}^{\infty}\frac{1}{2^k}\frac{p_k(x)}{1+p_k(x)} \qquad \textit{for all } x\in X.$$

Then:

(a) p is a paranorm on X and satisfies

(1.3.3.2) $p\ (x_n)\to 0\ (n\to\infty)$ if and only if $p_k\ (x_n)\to 0\ (n\to\infty)$ for each k;

(b) p is the weakest paranorm which is stronger than every p_k,

(c) p is total if and only if every p_k is total.

Example 1.3.4. The set $\mathbb{C}$ of complex numbers with the usual algebraic operations and $p=|\cdot|$, the modulus, is a totally paranomed space. If we put $d(z,\omega)=|z-\omega|$ for all $z,\omega\in\mathbb{C}$, then $(\mathbb{C},d)$ is a Frechet space.

By ω, we denote the set of all complex sequences $x=(x_k)_{k=0}^{\infty}$ which becomes a linear space with $x+y=(x_k+y_k)_{k=0}^{\infty}$ and $\lambda x=(\lambda x_k)_{k=0}^{\infty}$ for all $x,y\in\omega$ and $\lambda\in\mathbb{C}$

Theorem 1.3.5. The set ω is a Frechet space with respect to the metric d defined by

$$d(x,y)=\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{|x_k-y_k|}{1+|x_k-y_k|} \qquad \text{for all } x,y\in\omega.$$

Furthermore convergence in (ω,d) and coordinatewise convergence are equivalent that is $x^{(n)}\to x$ $(n\to\infty)$ in (ω,d) if and only if $x_k^{(n)}\to x_k(n\to\infty)$ for every k.

1.4. Schauder-Basis

Definition 1.4.1. A <u>*Schauder basis*</u> of a linear metric space X is a sequence (b_n) of vectors such that for each vector $x\in p$ there is a unique sequence (λ_n) of scalars with $\sum_{n=1}^{\infty}\lambda_n b_n=x$, that is $\lim_{m\to\infty}\sum_{n=1}^{m}\lambda_n b_n=x$.

For finite dimensional spaces the concepts of Schauder and algebraic bases coincide. In most cases of interest, however the concepts differ. Every linear space has an algebraic basis. But there are linear metric spaces without a Schauder basis, as we shall see later in this subsection.

Example 1.4.2. For each n=0,1,........, let $e^{(n)}$ be the sequence with $e_n^{(n)}=1$ and $e_k^{(n)}=0$ for $k\neq n$. Then $(e^{(n)})_{n=0}^{\infty}$ is a Schauder basis of ω. More precisely, every sequence

$x=(x_k)_{k=0}^{\infty}\in\omega$ has a unique representation $x=\sum_{k=0}^{\infty} x_k e^{(k)}$ that is $\lim_{m\to\infty} x^{[m]}=x$ for $x^{[m]}=\sum_{k=0}^{m} x_k e^{(k)}$, the m-section of x.

Theorem 1.4.3. Every complex linear metric space X with Schauder basis is separable.

Example 1.4.4. The set $l_\infty=\{x\in\omega:\sup_k|x_k|<\infty\}$ of all bounded sequences is a Banach space with $\|x\|_\infty=\sup_k|x_k|\ (x\in l_\infty)$ which has no schauder basis.

The following result gives the algebraic and topological properties of the sets l_∞, c, c_0 and l_p.

Theorem 1.4.5. (a) Each of the sets l_∞, c_0 and c is a Banach space with $\|\cdot\|_\infty$ defined by $\|x\|_\infty=\sup_k|x_k|$. Moreover $|x_k|\leq\|x\|_\infty$ for all $k=0,1,\ldots\ldots$

(b) The sets lp are Banach spaces for $1\leq p<\infty$ with $\|\cdot\|_p$ defined by $\|x\|_p=\left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p}$. Moreover $|x_k|<\|x\|_p$ for all $k=0,1,\ldots\ldots$

(c) The sequence $(e^{(n)})_{n=0}^{\infty}$ is a Schauder basis for each of the spaces c_0 and l_p for $1\leq p<\infty$. More precisely, every sequence $x=(x_n)_{n=0}^{\infty}$ in any of these spaces has a unique representation $x=\sum_{n=0}^{\infty} x_n e^{(n)}$.

(d) Let e be the sequences with $e_k=1$ for all $k=0,1,....$, we put $b^{(0)} = e$ and $b^{(n)} = e^{(n-1)}$ for n=1,2,.....Then the sequence $(b^{(n)})_{n=0}^{\infty}$ is a Schauder basis for c. More precisely, every sequence $x=(x_n)_{n=0}^{\infty} \in c$ has a unique representation $x = le + \sum_{n=0}^{\infty}(x_n - l)e^{(n)}$ where $l = l(x) = \lim_{n\to\infty} x_n$.

(e) The space l_∞ has no Schauder basis..

1.5. Some Topological Concepts

Some useful Definitions and Results 1.5.1. If S and M are subsets of a metric space (X,d) and $\in > 0$, then the set S is called *ε-net* of M if for any $x \in M$ there exists $s \in S$, such that $d(x,s)<\varepsilon$. If the set S is finite, then the $\varepsilon-net$ S of M is called finite *ε-net*. The set M is said to be *totally bounded* if it has a finite $\varepsilon-net$ for every ε>0. A subset M of a metric space X is Compact if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of that subsequence is in M. The set M is said to be *relatively compact* if the closure $\overline{M}$ of M is a compact set. If the set M is relatively compact then, M is *totally bounded*. If the metric space (X,d) is complete, then the set M is relatively compact if and only if it is totally bounded. A subset M of a metric space X is relatively compact if and only if every sequence (x_n) in M has a convergent subsequence; in that case the limit of that subsequence need not be in M.

If $x \in X$ and r>0 then the <u>*open ball*</u> with centre at x and radius r is denoted by B(x,r), $B(x,r)=\{y \in X : d(x,y)<r\}$. If X is a normed space, then we denote by B_X the <u>*closed unit ball*</u> in X and by S_X the <u>*unit sphere*</u> in X. Let M_X (or simply M) be the set of all nonempty and bounded subsets of a metric space (X,d). and let M_X^c (or Simply M^c) be the subfamily of M_X , consisting of all closed sets. Further, let N_X (or simply N) be the set of all non empty and relatively compact subsets of (X,d). Let $d_H : M_X x M_X \to \mathbb{R}$ be the function defined by

$$(1.5.1.1) \qquad d_H(S,Q)=\max\left\{\sup_{x\in S} d(x,Q), \sup_{y\in Q} d(y,S)\right\} \qquad (S,Q \in M_X).$$

The function d_H is called <u>*Hausdorff distance*</u> and $d_H(S,Q)$ $(S,Q \in M_X)$ is the Hausdorff distance of sets S and Q.

Let us remark that if $\phi \neq F \subset X, r>0$ and

$$B(F,r)=\bigcup_{x\in F} B(x,r)=\{y \in X : d(y,F)<r\}$$

is the open ball with centre in F and radius r, then (1.5.1.1) is equivalent to

$$d_H(S,Q)=\inf\{\in>0 : S \subset B(Q,\in) \text{ and } Q \subset k(S,\in)\}, \ (S,Q \in M_X).$$

It is well known that (M_X, d_H) is a *pseudometric space* and that (M^c_X, d_H) is a *metric space*.

Let X and Y be infinite-dimensional complex Banach spaces and denote the set of *bounded linear operators* from X in to Y by $B(X,Y)$. We put $B(X) = B(X,X)$. For T in B(X,Y), N(T) and R(T) will denote, respectively, the null space and the range space of T. A linear operator $L:X\to Y$ is called *compact* (or completely continous) if $D(L) = X$ for the domain of L, and for every sequence $\{x_n\}\subset X$ such that $||x_n||<C$, the sequence $\{L(x_n)\}$ has subsequence which converges in Y. A compact operator is bounded. An operator L in $B(X,Y)$ is of finite rank if $\dim R(L)<\infty$. An operator of finite rank is clearly compact. Let F(X,Y), $K(X,Y)$ denote the set of all finite rank and compact operators from X to Y, respectively. Set F(X) = F(X,X) and K(X) = K(X,X).

Let X be a vector space over a field $\mathbb{F}$. A subset E of X is said to be *convex* if $\lambda x+(\lambda\text{-}l)\, y\in E$ for all $x,y\in E$ and for all $\lambda\in(0,1)$.

Clearly the intersection of any family of convex sets is a convex set. If F is a subset of X, then the intersection of all convex sets that contains F is called *convex cover or convex hull of* F denoted by co(F).

The vector subspace lin$F$ is the set of all linear combinations of elements in F. We shall prove that there is an analogous representation of the set co(F). Let us mention that a convex combination of elements of the set F is an element of the form,

$$\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3+\ldots+\lambda_n x_n \ (x_i \in F, \lambda_i \geq 0)(\text{i}=1\ldots.\text{n}),\ \sum_{i=1}^{n} \lambda i=1\ (n\in \mathbb{N})).$$

Theorem 1.5.2. If X is a vector space over the field $\mathbb{F}$ and E,E_1,E_2,......, En are convex subsets of X and F⊂X, then

(1.5.2.1) $$\text{cvx (E)} \subset \text{E},$$

(1.5.2.2) $$co\ (\text{F}) = \text{cvx(F)}$$

(1.5.2.3) $$\text{co}\left(\bigcup_{i=1}^{n} E_i\right)=\left\{\sum_{i=1}^{n} \lambda_i E_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\right\}.$$

Proof. To prove (1.5.2.1) it suffices to show that for any $n\geq 2$.

(1.5.2.4) $x_i \in E, \lambda_i \geq 0$ (I=1,2.......n) and together

imply $\lambda_1 x_1 + \ldots + \lambda_n x_n \in E$.

We shall use the method of mathematical induction. For n=2 the statement clearly is true. Suppose that the statement in (1.5.2.4) is true for natural number n>2, and let us prove the statement for $n+1$. If $x_i \in E$, $\lambda_i \geq 0$ (i=1,2....., n+1) and $\sum_{i=1}^{n+1} \lambda_i = 1$, then there are two cases: first, if $\sum_{i=1}^{n}\lambda_i = 0$, then $\lambda_i = 0$ $(i=1,\dots.n)$ and $\lambda_1 x_1+\dots+\lambda_{n+1}\ x_{n+1} = X_{n+1} \in E$; second if $\lambda = \sum_{i=1}^{n}\lambda_i \neq 0$, then

$\lambda_1 x_1+\dots+\lambda_{n+1}\ x_{n+1} = \lambda(\lambda_1\lambda^{-1}x_1+\dots+\lambda_n\lambda^{-1}x_n)+\lambda_{n+1}x_{n+1} \in E$. Thus we have shown inclusion (1.5.2.1).

It follows from (1.5.2.1) that cvx(F) $\subset$co(F). Hence, since co(F) is a convex subset of X, it suffices to show that cvx(F) is convex. Suppose that $\lambda \in (0,1)$, and x,y$\in$ cvx(F). Then there exist $n, m \in \mathbb{N}$, α_i, x_i $(i=1,\dots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$, β_j, y_j $(j=1,\dots,m)$ with $\sum_{j==1}^{m}\beta_j = 1$ such that $x = \sum_{i=1}^{n}\alpha_i x_i$ and $y = \sum_{j=1}^{m}\beta_j y_j$. Now $\sum_{i=1}^{n}\lambda\alpha_i + \sum_{j=1}^{m}(1-\lambda)\beta_j = \lambda+(1-\lambda) = 1$ implies $\lambda x+(1-\lambda)y \in$ cvx(F).

Hence we have proved (1.5.2.2).

We put $S = \left\{\sum_{i=1}^{n}\lambda_i E_i : \lambda_i \geq 0, (i=1,\dots,n) \sum_{i=1}^{n}\ \lambda_i = 1\right\}$.By (1.5.2.2) it follows that $S \subset co\left(\bigcup_{i=1}^{n} E_i\right)$. Since $\bigcup_{i=1}^{n} E_i \subset S$, to prove (1.5.2.3), it suffices to show that S is convex. Suppose that $\lambda \in (0,1)$ and $x, y \in S$. Now there exist $\alpha_i, x_i (i=1,\dots n)$ with $\sum_{i=1}^{n}\alpha_i = 1, \beta_i, y_i (i=1,\dots,n)$ with $\sum_{i=1}^{n}\beta_i = 1$ such that $x = \sum_{i=1}^{n}\alpha_i x_i, y = \sum_{i=1}^{n}\beta_i y_i$. We put

$\gamma_i = \lambda\alpha_i + (1-\lambda)\beta_i (i=1,...n)$. Since E_1,... E_n, are convex, there exist $z_i \in E_i (i=1,...n)$ such that

(1.5.2.5) $$\lambda\alpha_i x_i + (1-\lambda)\beta_i y_i = \gamma_i z_i \quad \text{for } i=1,\dots,n.$$

Let us remark

(1.5.2.6) $$\sum_{i=1}^{n}\gamma_i = \lambda\sum_{i=1}^{n}\alpha_i + (1-\lambda)\sum_{i=1}^{n}\beta_i = \lambda + (1-\lambda) = 1.$$

By (1.5.2.5) and (1.5.2.6) we have $\lambda x + (1-\lambda)y = \sum_{i=1}^{n}\gamma_i z_i \in S$.

We continue with the study of convex sets in normed spaces.

Lemma 1.5.3. Let Q be a bounded subset of normed space X. Then for any $x \in X$.

(1.5.3.1) $$\sup_{y\in co(Q)} \| x-y \| = \sup_{z\in Q} \| x-z \|.$$

Corollary 1.5.4. Let Q be a bounded subset of a normed space X. Then the sets Q and co(Q) have equal diameter, that is diam(Q) = diam(co(Q)).

Let Q be a nonempty and bounded subset of a normed space X. Then the convex closure of Q, is denoted by Conv(Q), and Conv(Q) is the smallest convex and closed subset of X that contains Q. It is easy to prove that Conv(Q) = $\overline{\text{co}(Q)}$.

Corollary 1.5.5. Let Q be a bounded subset of a normed space X. Then the sets Q and Conv(Q) have equal diameters, that is diam(Q) = diam(Conv(Q)).

CHAPTER II

FK- SPACES

2.1. Introduction

Given any two subsets X and Y of ω and any infinite matrix $A=(a_{nk})_{n,k=0}^{\infty}$ of complex numbers, we shall write $A_n=(a_{nk})_{k=0}^{\infty}$ for the sequence in the n-th row of A,

$$A_n(x)=\sum_{k=0}^{\infty}a_{nk}x_k \qquad (x\in X)\text{ for all n=0,1,....,}$$

(provided the series converge) and

$$A(x)=(A_n(x))_{n=0}^{\infty}.$$

Furthermore let (X,Y) be the class of all matrices A that map X in to Y, that is for which the series $A_n(x)$ converge for all $x\in X$ and for all n, and $A(x)\in Y$ for all $x\in X$.

If $X\subset\omega$ is a linear metric space with respect to d_X and $a, x_0\in X$, then we shall write

$$S_\delta[x_0]=S_{X,\delta}[x_0]=\{x\in X: d_X(x,x_0)\leq\delta\} \quad (\delta>0)$$

$$\|a\|_D^*=\|a\|_{X,D}^*=\sup\left\{\left|\sum_{k=0}^{\infty}a_kx_k\right| : x\in S_{1/D}[0]\right\} \quad (D>0)$$

provided the expression on the right exists and is finite. By **Remark 1.4.1**. This is the case whenever X is an FK space and the series $\sum_{k=0}^{\infty} a_k x_k$ converge for all $x \in X$. If X is a BK space we write

$$\| a \|^* = \| a \|_X^* = \sup\left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| : \| x \| = 1 \right\}.$$

Let A be an infinite matrix, D a positive real and X an FK space. Then we put

$$M_{A,D}^*(X, l_\infty) = \sup_n \| A_n \|_D^*$$

and, if X is a BK space, then we write

$$M_A^*(X, l_\infty) = \sup_n \| A_n \|^*.$$

In this chapter we shall give an introduction in to the general theory of FK-spaces. It is the most powerful tool for the solution of problems of various kinds in summability, in particular in the characterization of matrix transformation between sequence spaces.

Finally we shall study the matrix transformation between some classical sequence spaces by giving necessary and sufficient condition on the entries of a matrix to belong to the respective class.

Most of the results used here are taken from Wilnasky [38].

2.2. Definition and Main Results

Definition 2.2.1. A Fréchet sequence space (X,d_X) is said to be an FK-Space if its metric d_X is stronger than the metric $d|_X$ of ω on X. A BK space is an FK space which is a Banach space.

Remark 2.2.2. By definition, an FK space X is continuously embedded in ω, that is the inclusion map $i:(X,d_X)\to(\omega,d)$ defined by $i(x)=x$ $(x\in X)$ is continuous. An FK space X is a Fréchet sequence space with continuous coordinates $P_k: X\to\mathbb{C}$ defined by $P_k(x)=x_k(k=0,1....)$ for all $x\in X$.

Example 2.2.3. The space ω is an FK space with its natural metric d. The spaces l_∞, c, c_0 and l_p $(1\leq p<\infty)$ are BK spaces with their natural norms.

Theorem. 2.2.4. Let (X,d_X) be a Fréchet space, (Y,d_Y) an FK space and $f:X\to Y$ a linear map. Then $f:(X,d_X)\to(Y,d|_Y)$ is continuous if and only if $f:(X,d_X)\to(Y,d_Y)$ is continuous.

Proof. First we assume that $f:(X,d_X)\to(Y,d_Y)$is continuous. Since Y is an FK space its metric d_Y is stronger than the metric $d|_Y$ of ω on Y. so $f:(X,d_X)\to(Y,d|_Y)$is continuous.

Conversely we assume that $f:(X,d_X)\to(Y,d|_Y)$ is continuous. Since $(Y,d|_Y)$ is a Hausdorff space and f is continuous, the graph of f, graph (f) = $\{(x,f(x)):\ x\in X\}$, is a closed set in $(X,d_X)\mathrm{x}(Y,d|_Y)$ by the closed graph lemma, hence a closed set in (X,d_X) x (Y,d_Y), since the FK metric d_Y is stronger than $d|_Y$. By the closed graph theorem, the map $f:(X,d_X)\to(Y,d_Y)$ is continuous.

Corollary 2.2.5. Let X be a Fréchet space, Y an FK space, $f:X\to Y$ a linear map and P_n the n-th co-ordinate, that is $P_n(y)= y_n$ $(y\in Y)$ for all n=0,1....... If each map $P_n\circ f:X\to\mathbb{C}$ is continuous, so is $f:X\to Y$.

Proof. Since $P_n\circ f:X\to\mathbb{C}$ is continuous for each n, the map $f:X\to\omega$ is continuous by the equivalence of coordinatewise convergence and convergence in ω. By Theorem 2.2.4, $f:X\to Y$ is continuous.

Remark 2.2.6. Let $X\supset\phi$ be an FK space and $a\in\omega$. If the series $\sum_{k=0}^{\infty}a_kx_k$ converges for each $x\in X$, then the linear functional $f_a:X\to\mathbb{C}$ defined by

$$f_a(x)=\sum_{k=0}^{\infty}a_kx_k \qquad \text{for all } x\in X$$

is continuous.

Proof. For each $n \in \mathbb{N}_0$, we define the linear functional $f_{a,n}: X \to \mathbb{C}$ by $f_{a,n}(x) = \sum_{k=0}^{n} a_k x_k$ for all $x \in X$. Since X is an FK space, the coordinates $P_k : X \to \mathbb{C}$ are continuous on X for all k=0,1..., and so are the functionals $f_{a,n} = \sum_{k=0}^{n} a_k P_k$ (n=0,1...). For each $x \in X, f_a(x) = \lim_{n\to\infty} f_{a,n}(x)$ exists, and so $f_a : X \to \mathbb{C}$ is continuous by the Banach-Steinhaus theorem.

Theorem 2.2.7. Any matrix map between FK spaces is continuous.

Proof. Let X and Y be FK spaces, $A \in (X,Y)$ and the map $f_A : X \to Y$ be defined by $f_A(x) = A(x)$ for all $x \in X$. Since the maps $P_n \circ f_A : X \to \mathbb{C}$ are continuous for all $n \in \mathbb{N}_0$ by Remark 2.2.6, the linear map f_A is continuous by corollary 2.2.5.

Definition 2.2.8. An FK space $X \supset \phi$ has <u>AK</u> if, for every sequence $x = (x_k)_{k=0}^{\infty} \in X$,

$$x = \sum_{k=0}^{\infty} x_k e^{(k)}, \text{ that is } x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x (m \to \infty),$$

and X has <u>AD</u> if ϕ is dense in X. If an FK space has AK or AD we also say that it is an <u>AK</u> or <u>AD</u> space.

Remark 2.2.9. Every AK space has AD. The converse is not true in general.

Proof. The first part is trivial, and the second part can be found in Wilansky [38, Example 5.2.5, p.78]

Example 2.2.10. The spaces ω, c_0 and l_p ($1\leq p<\infty$) all have AK by Example 1.4.2 and Theorem 1.4.5.

Note that the FK metric of an FK space will turn out to be unique.

Theorem 2.2.11. Let X and Y be FK spaces and $X\subset Y$. Then the metric d_X on X is stronger than the metric $d_Y|_X$ of Y on X. The metrics are equivalent if and only if X is a closed subspace of Y. In particular, the metric of an FK space is unique, this means there is at most one way to make a linear subspace of ω in to an FK space.

Proof. Let $i:(X,d_X)\to(Y,d_Y)$ be the inclusion map. Since X is an FK space, $i:(X,d_X)\to(Y,d|_Y)$ is continuous, and so is $i:(X,d_X)\to(Y,d_Y)$ by Theorem 2.2.4. Thus d_X is stronger than $d_{Y|X}$. The uniqueness of an FK space is shown in exactly the same way. Let X be closed in Y, then X becomes an FK space with $d_Y|_X$, and the uniqueness of an FK metric implies that d_X and $d_Y|_X$ are equivalent.

Conversely, if d_X and $d_Y|_X$ are equivalent, then X is complete subspace of Y, hence a closed subspace of Y.

Example 2.2.12. The BK spaces c_0, and c are closed subspaces of l_∞. Thus the BK norms on c_0,c and l_∞ must be the same. The BK space l_1 is a subspace of l_∞ which is not closed in l_∞. Thus its BK norm $\|\cdot\|_1$ is strictly stronger than the BK norm $\|\cdot\|_\infty$ on l_∞.

2.3. Matrix transformation into l_∞, c and c_0

In this section we shall apply the results of section 2.2 to characterize classes (X,Y) where X is any FK space and Y is any of the spaces l_∞, c and c_0.

Theorem 2.3.1. Let X and Y be FK spaces.

(a) Then $(X,Y) \subset B(X,Y)$, that is, every $A \in (X,Y)$ defines a liner operator $L_A \in B(X,Y)$ where $L_A(x) = A(x)$ for all $x \in X$.

(b) Then $A \in (X, l_\infty)$ if and only if.

(2.3.1.1) $\qquad \|A\|_D^* = M_{A,D}^*(X,l_\infty) < \infty \qquad$ for some D>0.

If X is a BK space and $A \in (X,l_\infty)$, then $\|A\|^* = M_A^*(X,l_\infty) = \|L_A\| < \infty$.

(c) If $(b^k)_{k=0}^\infty$ is Schauder basis for X, and Y_1 a closed FK space in Y, then $A \in (X,Y_1)$ if and only if $A \in (X,Y)$ and $A(b^{(k)}) \in Y_1$ for all $k=0,1,\ldots$.

Proof. Part (a) is Theorem 2.2.7. (b) First we assume that condition (2.3.1.1) holds. Then, for all $x \in S_{1/D}[0]$, the series $A_n(x)$ ($n=0,1,\ldots$) converge and $A(x) \in l_\infty$. Since the set $S_{1/D}[0]$ is absorbing by let (X,p) be a paranormed space. Then the open neighbourhoods of $0, N_r(0) = \{x \in X : p(x) < r\}$, are absorbing for all r>0, we conclude that $A_n(x)$ converges for each $x \in X$ and $A(x) \in l_\infty$ for all $x \in X$, hence $A \in (X,l_\infty)$.

Conversely, we assume $A \in (X,l_\infty)$. Then L_A is continuous by part (a). Hence there exist a neighbourhood N of 0 in X and a real D>0 such that $S_{1/D}[0] \subset N$ and $\|L_A(x)\| < 1$ for all $x \in N$. This implies condition (2.3.1.1). If X is a BK space, then $L_A \in B(X,Y)$ implies

$$\| A(x) \|_\infty = \sup_n | A_n(x) | = \| L_A(x) \|_\infty \leq \| L_A \| \text{ for all } x \in X \text{ with } \|x\|=1.$$

Thus $| A_n(x) | \leq \| L_A \|$ for all n and for all $x \in X$ with $\|x\|=1$ and, by the definition of the norm $\| \cdot \|^*$,

$$\| A \|^* = \sup_n \| A_n \|^* \leq \| L_A \|. \tag{2.3.1.2.}$$

Further, given $\varepsilon > 0$, there is $x \in X$ with $\|x\|=1$ such that $\| A(x) \|_\infty \geq \| L_A \| - \varepsilon/2$, and there is $n(x) \in N_0$ with $| A_{n(x)}(x) | > \| A(x) \|_\infty - \varepsilon/2$, consequently $| A_{n(x)}(x) | \geq \| L_A \| - \varepsilon$. Therefore $\| A \|^* = \sup_n \| A_n \|^* \geq \| L_A \| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\| A \|^* \geq \| L_A \|$, and, with (2.3.1.2.), we have $\| A \|^* = \| L_A \|$.

(c)The necessity of the conditions for $A \in (X, Y_1)$ is trivial.

Conversely, if $A \in (A, Y)$, then $L_A \in B(X, Y)$. Since Y_1 is a closed subspace of Y, the FK metrics of Y_1 and Y are the same by Theorem 2.2.11. Consequently, if S is any subset in Y_1, then, for its closures $\operatorname{clos}_{Y_1}(S)$ and $\operatorname{clos}_{Y|Y_1}(S)$ with respect to the metrics d_{Y_1} and $d_{Y|Y_1}$, we have

$$\operatorname{clos}_{Y_1}(S) = \operatorname{clos}_{Y|Y_1}(S). \tag{2.3.1.3}$$

Let $x \in X$ and $SB = \{\sum_{k=0}^m \lambda_k b^{(k)} : m \in \mathbb{N}_0, \lambda_k \in \mathbb{C} (k = 0,1,\ldots)$ denote the span of $\{b^{(k)} : k = 0,1,\ldots$. Since $L_A(b^{(k)}) \in Y_1$ for all k=0,1,... and the metrics d_{Y_1} *and* $d_{Y|_{Y_1}}$ are equivalent, the map $L_A|_{SB}:(X,d_X)\to(Y_1,d_{Y1})$ is continuous. Further, since $(b^{(k)})_{k=0}^{\infty}$ is a basis of X, we have $\overline{SB}=X$. Therefore, by (2.3.1.3.) and the continuity of $L_A|_{SB}$, we have

$$L_A(X)=L_A(\overline{SB})=clos_{Y1}(L_A|_{SB}(SB))=clos_{Y|_{Y_1}}(L_A|_{SB}(SB))$$

$$\subset clos_{Y|_{Y_1}}(Y_1)=Y_1$$

Thus $A(x)\in Y$ for all $x\in X$.

2.4. Matrix transformations between some classical sequence spaces

Let A be an infinite matrix. We write $q=p/(p-1)$ for $1<p<\infty$, $q=\infty$ for $p=1$ and $q=1$ for $p=\infty$, put

$$M_A(l_p,l_\infty)=\begin{cases}\|A\|_\infty=\sup_{n,k}|a_{nk}| & (p=1)\\ \|A\|_q=\sup_p\left(\sum_{k=0}^{\infty}|a_{nk}|^q\right) & (1<p\leq\infty)\end{cases}$$

and consider the conditions

(2.4. (a)) $\lim_{n\to\infty} a_{nk}=0 \qquad (k=0,1,\ldots),$

(2.4.(b)) $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}a_{nk}\right)=0,$

(2.4.(c)) $\lim_{n\to\infty} a_{nk}=l_k$ for some $l_k\in\mathbb{C}(k=0,1,\ldots)$

(2.4.(d)) $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty} a_{nk}\right)=l$ for some $l\in\mathbb{C}$.

Theorem 2.4.1. We have

(2.4.1.1.) $(c_0,l_\infty)=(c,l_\infty)=(l_\infty,l_\infty)$ and $A\in(l_\infty,l_\infty)$ if and only if

$$(*)\ldots\ldots\ldots\ldots\ldots\ldots M_A(l_\infty,l_\infty)=\sup_n\left(\sum_{k=0}^{\infty}|a_{nk}|\right)<\infty,$$

(2.4.1.2) $A\in(c_0,c_0)$ if and only if conditions (*) and (2.4.(a)) hold;

(2.4.1.3) $A\in(c,c_0)$ if and only if conditions (*), (2.4(a)) and (2.4.(b)) hold;

(2.4.1.4) $A\in(c_0,c)$ if and only if conditions (*) and (2.4.(c)) hold;

(2.4.1.5) $A\in(c,c)$ if and only if conditions (*), (2.4(c)) and (2.4(d)) hold.

Proof. (2.4.1.1.) We have $A\in(l_\infty,l_\infty)$ if and only if condition (*) holds by Theorem 2.3.1 and 3.3.1.

Further, if condition (*) holds, then $A\in(l_\infty,l_\infty)\subset(c_0,c)$, since $c_0\subset l_\infty$.

Conversely, let $A \in (c_0, l_\infty)$. Then $\sup_n \| A_n \|_{c_0}^* < \infty$ by Theorem 2.3.1(b). Since the series $A_n(x)$ converge for all x and n, we have $f_{A_n} \in c_0^*$ for all n where $f_{A_n}(x) = \sum_{k=0}^{\infty} a_{nk} x_k$ for all $x \in c_0$, hence $| f_{A_n}(x) | \leq \| f_{A_n} \| = \| A_n \|_{c_0}^*$. We fix $n \in \mathbb{N}_0$. Let $m \in \mathbb{N}_0$ be arbitrary. We define the sequence $x^{[m,n]}$ by $x^{[m,n]} = \sum_{k=0}^{m} \operatorname{sgn}(a_n k) e^{(k)}$. Then $x^{[m,n]} \in c_0, \| x^{m,n]} \|_\infty \leq 1 and | f_{A_n}(x^{[m,n]}) | = \sum_{k=0}^{m} | a_{nk} | \leq \| A_n \|_{c_0}^*$. Since $m \in \mathbb{N}_0$ was arbitrary, $\| A_n \|_1 = \sum_{k=0}^{\infty} | a_{nk} | \leq \| A_n \|_{c_0}^*$ for all n=0,1,.... Therefore condition (*) must hold. Finally $c_0 \subset c \subset l_\infty$ and $(c_0, l_\infty) = (l_\infty, l_\infty)$ together imply $(c, l_\infty) = (l_\infty, l_\infty)$.

Part (2.4.1.2) to (2.4.1.5) follow from part (2.4.1.1), Theorem 2.3.1(c) and Theorem 1.4.5.

Theorem 2.4.2. Let $1<p<\infty$. Then:

(a) $A \in (l_p, l_\infty)$ if and only if

(2.4.2.1) $$M(l_p, l_\infty) < \infty;$$

(b) $A \in (l_p, c_0)$ if and only if conditions (2.4.2.1) and (2.4.(a)) hold;

(c) $A \in (l_p, c)$ if and only if conditions (2.4.2.1) and (2.4(c)) hold.

CHAPTER III

DUALS OF SOME SEQUENCE SPACES

3.1. Introduction

In this chapter we shall study the α-, β- and continuous duals of some sets of sequences. The first two kinds of duals spaces naturally arise in the study of absolutely and ordinary convergence of sequences from a subset of ω.

Furthermore the conditions given in subsection 2.2 for on infinite matrix A to be in the classes (X, l_∞), (X, c) and (X, c_0) for arbitrary FK-spaces X involved the norm of the operator L_A defined by $L_A(x) = A(x)$. Since $A \in (X, Y)$ can only hold if $A_n(x) = \sum_{K=0}^{\infty} A_{nk} x_k$ converges for all $x \in X$, the so called β-dual of X. finally if X and Y are given FK-spaces, then we intend to replace the operator norm in the conditions for $A \in (X, Y)$ by conditions for the entries of the matrix A. In many cases this can be achieved by replacing the operator norm by the natural norm on the β-dual of X.

The α-and β-duals are special cases of the so-called multiplier spaces.

Also we shall give the continuous duals of the spaces l_p for $1 \leq p < \infty$, c and c_0.

Most of the results used here are taken from Wilansky [36], Wilansky [38], G. Köthe [16], W. Ruckle [33].

3.2. The α-and β-duals of sets of sequences

Definition 3.2.1. Let X and Y be subsets of ω.

(a)For all $z \in \omega$, we write $z^{-1} * Y = \{x \in \omega : xz = (x_k z_k)_{k=0}^{\infty} \in Y\}$.

The set Z=M(X,Y) = $\bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y$ for all $x \in X\}$ is called the multiplier space of X and Y.

(b) By cs and bs, we denote the set of all convergent and bounded series, respectively, that is $cs = \{x \in \omega: \sum_{k=0}^{\infty} x_k$ converges$\}$ and $bs = \{x \in \omega : \left(\sum_{k=0}^{n} x_k\right)_{k=0}^{\infty} \in l_\infty\}$ and we define the norm $\|.\|_{bs}$ on cs and bs by $\|x\|_{bs} = \sup_n |\sum_{k=0}^{n} x_k|$. In the special case where $Y = l_1$ or Y=cs, the multiplier spaces $X^\alpha = M(X, l_1)$ and $X^\beta = M(X, cs)$ are called the α-or köthe-Toeplitz and β-duals of X. If † denotes either of the symbols α or β, then $X \subset \omega$ is said to be †-perfect if $X^{\dagger\dagger} = (X^\dagger)^\dagger = X$.

Lemma 3.2.2. Let X,Y,Z$\subset\omega$ and $\{X_\delta : \delta \in A\}$ be any collection of subsets of ω. Then

(i) X$\subset$M (M(X,Y),Y)

(ii) X$\subset$Z implies M(Z,Y) $\subset$M(X,Y)

(iii) M(X,Y)=M(M(M(X,Y),Y),Y)

(iv) $M(\bigcup_{\delta\in A} X_\delta, Y) = \bigcap_{\delta\in A} M(X_\delta, Y)$.

As an immediate consequence of Lemma 3.2.2 we obtain

Corollary 3.2.3. Let X,Y $\subset\omega$ and $\{X_\delta : \delta \in A\}$ a collection of subsets of ω, if † denotes either of the symbols α or β then

(i) X$\subset$X†† (ii) X$\subset$Y implies Y$^\dagger\subset$ X†

(iii) X† = X††† (iv) $\left(\bigcup_{\delta\in A} X_\delta\right)^\dagger = \bigcap_{\delta\in A} X_\delta^\dagger$

A subset X of ω is said to be <u>normal</u> if $x \in X$ and $|\tilde{x}_k| \le |x_k|$ (k=0,1....) together imply $\tilde{x} \in X$.

Remark 3.2.4. Oviously X$^\alpha\subset$X$^\beta$ for arbitrary X$\subset\omega$. If X is normal subset of ω, then X$^\alpha$ = X$^\beta$

Proof. The first part is obvious, for the second part, we have to show that X$^\beta\subset$X$^\alpha$. Let $a \in X^\beta$ and $x \in X$ be given. We define the sequence y by y$_k$ = sgn (x_k)|x_k| for k = 0,1,2..... Then obviously |y$_k$|$\le$|x_k|for all k, and consequently $y \in X$, since X is normal, and so $ax \in cs$. Further by the definition of the sequence y, ay = $(|a_k||x_k|)_{k=0}^{\infty} = |ax| \in cs$, hence $ax \in l_1$. Since $x \in X$ was arbitrary, $a \in X^{\alpha}$. This shows $X^{\beta} \subset X^{\alpha}$.

Example 3.2.5. We have

(i) $M(c_0, c) = l_\infty$ (ii) $M(c,c) = c$ (iii) $M(l_\infty, c) = c_0$

Proof. (i) If $a \in l_\infty$, then $ax \in c$ for all $x \in c_0$ and so $l_\infty \subset M(c_0, c)$. Conversely we assume $a \notin l_\infty$. Then there is a subsequence $(a_{k_j})_{j=0}^{\infty}$ of the sequence a such that $|a_{k_j}| > j+1$ for all j=0,1,..... We define the sequence x by

(3.2.5.1) $$x_k = \begin{cases} (-1)^j / a_{k_j} & for\, k = k_j \\ 0 & for\, k \neq k_j \end{cases} \quad (j = 0,1,.........).$$

Then $x \in c_0$ and $a_{k_j} x_{k_j} = (-1)^j$ for all $j = 0,1,......$, hence $ax \notin c$. This shows $M(c_0, c) \subset l_\infty$.

(ii) If $a \in c$, then $ax \in c$ for all $x \in c$, and so $c \subset M(c,c)$.

Conversely we assume $a \notin c$. Since $e \in c$ and $ae = a \notin c$, we have $a \notin M(c,c)$. This shows $M(c,c) \subset c$.

(iii) If $a \in c_0$, then $ax \in c$ for all $x \in l_\infty$, and so $c_0 \subset M(l_\infty, c)$.

Conversely we assume that $a \notin c_0$. Then there are a real b>0 and a subsequence $(a_{k_j})_{j=0}^{\infty}$ of the sequence a such that $|a_{k_j}| > b$ for all j=0,1,...... We define the sequence x as in (3.2.5.1). Then

$x \in l_\infty$ and $a_{k_j}\, x_{k_j} = (-1)^j$ for j=0,1,.... hence $a \notin M(l_\infty, c)$. This shows $M(l_\infty, c) \subset c_0$.

Now we shall give the α-and β-duals of the classical sequence spaces.

Theorem 3.2.6. Let † denotes either of the symbols α or β. Then (a) $\omega^\dagger = \phi$ and $\phi^\dagger = \omega$.

(b) $l_1^\dagger = l_\infty$, $l^\dagger_p = lq$ for $1<p<\infty$ and $q = p/p-1$, and for all $a \in l^\beta_p$, $\| a \|^*_{l_1} = \| a \|_\infty, and \| a \|^*_{l_p} = \| a \|_q$ for $1<p<\infty$.

(c) $c_0^\dagger = c^\dagger = l_\infty^\dagger = l_1$ and $\| a \|^*_{c0} = \| a \|^*_c = \| a \|^*_{l_\infty} = \| a \|_1$ for all $a \in l^\beta_\infty$.

The multiplier space of two BK spaces will turn out to be a BK space.

Theorem 3.2.7. Let $(X, \|.\|_x)$ and $(Y, \|.\|_Y)$ be BK spaces with $X \supset \phi$ and $Z=M(X,Y)$. Then Z is a BK space with $\|.\|$ defined by

$$\| z \| = \| z \|^*_X = \sup\{\| xz \|_Y : \| x \|_X = 1\} \qquad \text{for all } z \in Z.$$

Proof. It is well known that B=B(X,Y) is a Banach-space. Each defines a diagonal matrix map $\hat{z} : X \to Y$ where $\hat{z}$ (x) $=xz$ for all $x \in X$ which is continuos by the Theorem 2.2.7. This embeds Z in B, for $\hat{z} = 0$, then $\hat{z}(e^{(n)}) = (z_n)_{n=0}^\infty = 0 = z$. To see that the coordinates are continuous, we fix $n \in \mathbb{N}_0$ and put $u=1/\|e^{(n)}\|_x$ and $v = \|e^n\|_Y$. Then $\|ue^{(n)}\|_x=1$ and

$uv|z_n| = u\,||z_n e^{(n)}||_Y = u||e^{(n)}z||_Y = ||(ue^{(n)})z||_Y \leq || z ||_X^* = ||z|| \ \forall n.$

It remains to show that Z is closed subspace of B. Let $(\hat{z}^{(m)})_{m=0}^{\infty}$ be a sequence in B with $\hat{z}^{(m)} \to T \in B$. For each fixed $x \in X$, we obtain $\hat{z}^{(m)}(x) \to T(n) \in Y (m \to \infty)$ and since Y is a BK-space, this implies $(\hat{z}^{(m)}(x_k))_k \to (T(x))_k$ that is $z_k^{(m)} x_k \to (T(x))_k$ (m→∞) for each fixed k. we put $x = e^{(k)}$. Then $z_k^{(m)} \to (T(e^k))_k = t_k$ (m→∞) and so $x_k z_k^{(m)} = (T(x))_k$ (m→∞) and $x_k z_k^{(m)} \to (T(x))_k$ (m→∞). Therefore $T(x) = xt$ for all $x \in X$, and so $T = \hat{t}$.

Corollary 3.2.8. The α-and β-duals of a *BK*-space X are BK spaces with respect to $||a||_\alpha = ||a||_{x,\alpha} = \sup \{ ||ax||_1 = \sum_{k=0}^{\infty} |a_k x_k| : || x || \leq 1 \}$ and $||a||_\beta = ||a||_{x,\beta} = \sup\{ ||(ax)||_{bs} = \sup_n |\sum_{k=0}^{n} a_k x_k| : || x || \leq 1 \}$.

Example 3.2.9. Let X be any of the spaces l_∞, c, c_0 and l_p for $1 \leq p < \infty$. Then the norms $\| \cdot \|_{X^\beta}, \| \cdot \|_X^*$, $\| \cdot \|_{X,\alpha}$ and $\| \cdot \|_{X,\beta}$ are equivalent on X^β.

Proof. The norm $\| \cdot \|_X^*$ and the natural norm $\| \cdot \|_{X^\beta}$, are equal on X^β by Theorem 3.2.6. Since each set X^β is a *BK*-space with its natural norm, $\| \cdot \|_{X^\beta}$ and $\| \cdot \|_{X,\beta}$ are equivalent by Corollary 3.2.8 and Theorem 2.2.11.

Finally since $X^{\alpha} = X^{\beta}$ for each set X the norms $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{X;\beta}$ are equivalent by Corollary 3.2.8 and Theorem 2.2.11.

The analogues of Theorem 3.2.6 and Corollary 3.2.7 do not hold for FK-spaces in general.

Remark 3.2.10. The space ω is an FK space and $\omega^{\alpha} = \omega^{\beta} = \phi$ and ϕ has no Frechet metric (cf.Wilansky [38]).

3.3. The continuos duals of the classical sequence spaces

There is a close relation between the β-dual and the continuos dual of an FK-space which is very useful in the determination of the continuos duals of the spaces l_{p},c and c_0.

Theorem 3.3.1. Let X be a BK-space and $X \supset \phi$. Then There is a linear one-to one map. $T:X^{\beta} \to X'$; we denote this by $X^{\beta} \subset X'$. If X has AK, then T is onto.

Proof. We define the map T on X^{β} as follows. For every $a \in X^{\beta}$, let $Ta:X \to C$ be defined by $(Ta)(x) = \sum_{k=0}^{\infty} a_k x_k$ for all $x \in X$.

Since $a \in X^{\beta}$, the series $\sum_{k=0}^{\infty} a_k x_k$ converges for all $x \in X$,and obviously T_a is linear. Further, since X is an FK space, $T_a \in X'$ for each $a \in X^{\beta}$. Therefore $T: X^{\beta} \to X'$. Further it is easy to see that T is linear.

To show that T is one-to-one, we assume $a,b \in X^{\beta}$ with $T_a = T_b$. This means $(T_a)(x) = (T_b)(x)$ for all $x \in X$. Since $\phi \subset X$, we may choose $x = e^{(k)}$ for each $k \in \mathbb{N}_0$ and obtain $(T_a)(e^{(k)}) = a_k = b_k = (T_b)(e^{(k)})$ for k=0,1,.... and so $a=b$.

Now we assume that X has AK and $f \in X'$. We put $a_n = f(e^{(n)})$ for n=0,1,..... Let $x \in X$ be given. Then $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, since X has AK, and $f \in X'$ implies $f(x) = \sum_{k=o}^{\infty} x_k f(e^{(k)}) = \sum_{k=0}^{\infty} a_k x_k = (T_a)(x)$, As $x \in X$ was arbitrary and the series converge, $a \in X^{\beta}$ and $f = T_a$. This shows that T is on to X'.

This completes the proof of the theorem.

Now we shall give the continuos duals of the classical sequence spaces.

Two linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are called <u>norm isomorphic</u> if there is an isomorphism $T: X \rightarrow Y$ such that $\|T(x)\|_Y = \|x\|_X$ for al $x \in X$; we shall write $X \simeq Y$.

Theorem 3.3.2. We have:

(a) $l_p^* \simeq l_\infty$ for $0<p \leq 1$ and $l_p^* \simeq l_q$ for $1<p<\infty$ where $q = p/p-1$;

(b) $c_0^* \simeq l_1$;

(c) $f \in c^*$ if and only if $f(x) = l\chi_f + \sum_{k=0}^{\infty} a_k x_k$ with $a \in l_1$ where $l = \lim_{k\to\infty} x_k$ and $\chi_f = f(e) - \sum_{k=0}^{\infty} a_k$. Furthermore $\| f \|^* = |\chi_f| + \| a \|_1$.

It is worth mentioning that the continuous dual of l_∞ is not isomorphic to a sequence space (cf.[16,31.1 pp. 427, 428] or [36, example 6.4.8, pp. 93, 94].

CHAPTER IV

MADDOX SEQUENCE SPACES AND THEIR DUALS

4.1. Introduction

If (X,g) is a paranormed space, with paranorm g (see Maddox [24]), then we denote by X^* the continuous dual of X, i.e. the set of all continuous linear functionals on X. If E is a set of complex sequences $x=(x_k)$ then E^t will denote the generalized Köthe-Toeplitz dual of E.

$$E^{\dagger} = \left\{ a : \sum_{1}^{\infty} a_k x_k \ \ converges,\ for\ \ all\ \ x \in E \right\}.$$

In Maddox [24] an attempt was made to determine the space $l^*(p)$, where

$$l(p) = \{x = (x_k) : \sum |x_k|^{p_k} < \infty\}, \quad 1 < p_k \le \sup p_k < \infty.$$

It was shown in Maddox [24] that l^*(p) could be identified with $l(q)$, where $(1/p_k)+(1/q_k) = 1$, provided $1<\inf p_k \le \sup p_k < \infty$. The restriction $\sup p_k < \infty$ is natural, in that it is necessary and sufficient for $l(p)$ to be a linear space (see Maddox [23]). If we write M=supp_k, then, as shown in Maddox [24] , the natural paranorm on $l(p)$ is

$$g(x) = (\sum |x_k|^{p_k})^{1/M}.$$

In this chapter we remove the restriction $\inf p_k > 1$ and determine $l^*(p)$ for any p such that $1 < p_k < \sup p_k < \infty$. First, however, we regard $l(p)$ merely as a set and suppose only that $1 < p_k$. Then we shall show that $l^{\dagger}(p)$ is the sequence space $M(p)$ defined as follows. We say that $a = (a_k) \in M(p)$ if and only if there exists an integer $N > 1$ such that $\sum |a_k|^{q_k} N^{-q_k/p_k} < \infty$,

Where we write $(1/p_k) + (1/q_k) = 1$.

Let us write S for the set of all k for which $0 < p_k \leq 1$ and $-S$ for the complement of S. Then we say that $a = (a_k) \in \hat{M}(p)$ if and only if

(i) $\sup_{k \in S} |a_k|^{p_k} < \infty$,

(ii) $\sum_{k \in -S} |a_k|^{q_k} N^{-q_k/p_k} < \infty$, for some integer $N > 1$.

If S is empty we regard (i) as being automatically satisfied; similarly if $-S$ is empty.

In $\hat{M}(p)$ we take the topology associated with the uniform convergence on the spheres of $l(p)$; precisely, we say that $a^{(n)} \to a$ in $\hat{M}(p)$ if and only if $(a^{(n)}, x) \to (a, x)$ uniformaly for x in any sphere of $l(p)$. Here we take the paranorm

$$g(x) = (\sum |x_k|^{p_k})^{1/M}$$

on $l(p)$, where M=max $(1, \sup p_k)$. By the argument of Theorem 4.2.2 we know that every $f \in l^*(p)$ is uniquely representable as $f(x) = \sum a_k x_k = (a,x)$, for every $x \in l(p)$, where $a \in \hat{M}(p)$. Hence we now have

In Maddox [24] the problem of characterizing the dual of the space $c_0(p)$ was posed. The solution is now given. First, we recall that $c_0(p)$ is the set of all sequences x such that $|x_k|^{p_k} \to 0$. The boundedness of (p_k) is necessary and sufficient for $c_0(p)$ to be a linear sequence space. The paranorm $g(x) = \sup_k |x_k|^{p_k/M}$, with $M = \max(1, \sup p_k)$, then makes $c_0(p)$ a topological linear space (see Maddox [24]). Now we introduce the space $M_0(p)$, which is the Köthe-Toeplitz dual of $c_0(p)$:

$$M_0(p) = \bigcup_{N>1} \{a : \sum |a_k| N^{-1/p_k} < \infty\}.$$

It is easy to check that $M_0(p)=l_1$ if and only if $\inf p_k>0$. We shall take in $c_0^*(p)$ the topology of uniform convergence on the spheres of $c_0(p)$.

Most of the concepts used here are taken from Maddox [23], [24], [26], [27], Simons [35], and Lascarides and Maddox [20].

4.2. Main Results

Theorem 4.2.1. Let $1<p_k$ for all k. Then $l^{\dagger}(p) = M(p)$.

Proof. Let $a \in M(p)$ and $x \in l(p)$. From the inequality

$$|b_k y_k| \leq |b_k|^{q_k} + |y_k|^{p_k}$$

we obtain $$|a_k x_k| \leq |a_k|^{q_k} N^{-q_k/p_k} + N|x_k|^{p_k},$$

where N is the integer associated with $a \in M(p)$. Hence $\sum a_k x_k$ is absolutely convergent and so $M(p) \subset l^{\dagger}(p)$.

Now let $a \in l^{\dagger}(p)$, so that $\sum a_k x_k$ converges for all $x \in l(p)$. This implies that $a \in M(p)$, for otherwise we can determine integers $0=n(0) < n(1) < n(2) < \ldots$ such that

$$M_s = \sum_{I(s)} |a|^q (s+1)^{-q/p} > 1$$

for $s = 1,2 \ldots$. For simplicity in notation we are omitting the k in a_k, p_k and q_k. The sum defining M_s is taken over k in the set $I(s) = \{k: n(s-1)+1 \leq k \leq n(s)\}$. Now for k in $I(s)$, define, again omitting k,

$$x = (\mathrm{sgn}\, a)|a|^{q-1}(s+1)^{-q} M_s^{-1}.$$

We then have $$\sum_{I(s)} a_k x_k = (s+1)^{-1},$$

and $$\sum_{I(s)} |x_k|^{p_k} = \sum_{I(s)} |a|^q (s+1)^{-p-q} M_s^{-p}$$

$$\leq M_s^{-1}(s+1)^{-1} \sum_{I(s)} |a|^q (s+1)^{-q}$$

$$= M_s^{-1}(s+1)^{-2} \sum_{I(s)} |a|^q (s+1)^{-q/p}$$

$$= (s+1)^{-2}.$$

Thus $x \in l(p)$ but $\sum a_k x_k$ diverges, whence we must have $a \in M(p)$.

It is now easy to deduce from Theorem 4.2.1 that the continuous dual of $l(p)$ is, algebraically, $M(p)$.

Theorem 4.2.2. Let $1 < p_k \leq \sup p_k < \infty$, for all k. Then $l^*(p)$ is isomorphic to $M(p)$.

Proof. Write the unit vectors in $l(p)$ as e_k, $k = 1, 2, \ldots$ Then $x = \sum x_k e_k$ for every x in $l(p)$, whence $f(x) = \sum a_k e_k$ for any f in $l^*(p)$, where $f(e_k) = a_k$. By Theorem 4.2.1, the convergence of $\sum a_k x_k$ for every x in $l(p)$ implies that $a \in M(p)$.

Now if $x \in l(p)$ and we take any $a \in M(p)$, then $\sum a_k x_k$ converges, by Theorem 4.2.1., and clearly defines a linear functional on $l(p)$. Using the argument of Theorem 4.2.1 it is easy to check that

$$(4.2.2.1) \qquad \left|\sum a_k x_k\right| \leq \left(\sum |a_k|^{q_k} N^{-q_k/p_k} + N\right) g(x),$$

whenever $g(x) \leq 1$. Hence $\sum a_k x_k$ defines an element of $l^*(p)$. It is now evident that the map $T: l^*(p) \to M(p)$, given by $T(f) = a$, is a linear bijection.

We now wish to give a fuller identification of $l^*(p)$. Topologies will be defined on $l^*(p)$ and $M(p)$, in such a way that these spaces become linearly homeomorphic. In the case when $1<\inf p_k \leq \sup p_k < \infty$, so that $\sup q_k < \infty$, we shall show that $l^*(p)$ is linearly homeomorphic to $l(q)$, equipped with the natural paranorm

$$\left(\sum |x_k|^{q_k}\right)^{1/H},$$

where $H = \sup q_k$ and $x \in l(q)$.

First, we observe that $M(p) = l(q)$ if and only if $p = \inf p_k > 1$. For $p > 1$ implies for every $N > 1$, $1 < N^{q_k/p_k} \leq N^{1/(p-1)}$, whence $M(p) = l(q)$. Conversely, if $M(p) = l(q)$ but $\inf p_k = 1$, then there exist $k_1 < k_2 < \ldots$ such that $p_{k_n} < 1+1/n$. Define $x_k = 1$ for $k = k_n$ and $x_k = 0$ otherwise. Then

$$\sum \frac{|x_k|^{q_k}}{2^{q_k/p_k}} < 1,$$

so that $x \in M(p)\text{-}l(q)$. Hence we must have $p>1$.

Now we give topologies in $l^*(p)$ and $M(p)$. In $l^*(p)$ we employ the topology of uniform convergence on the spheres of $l(p)$; by sphere we shall mean sphere centred at the origin. Thus, $f_n \to f$ in $l^*(p)$ means that $f_n(x) \to f(x)$ uniformly for x in any sphere of $l(p)$.

Let us write

$$(a,x) = \Sigma a_k x_k$$

for $a \in M(p)$ and $x \in l(p)$. Then we define $a^{(n)} \to a$ in $M(p)$ to mean $(a^{(n)},x) \to (a,x)$ uniformly for x in any sphere of $l(p)$. With these definitions it is clear that $l^*(p)$ and $M(p)$ are topological linear spaces. Aslo, the map T of Theorem 4.2.2 is now seen to be a homeomorphism. Thus we have

Theorem 4.2.3. *If* $1<p_k \leq \sup p_k < \infty$, then $M(p)$ and $l^*(p)$ are linearly homeomorphic.

In the case $1 < \inf p_k$ when $M(p) = l(q)$, we also have.

Theorem : 4.2.4. If $1<\inf p_k \leq \sup p_k < \infty$ and $l(q)$ has its natural paranorm topology, then, $l^*(p)$ is linearly homeomorphic to $l(q)$.

Proof. Let us write

$$h(a) = \left(\sum |a_k|^{q_k}\right)^{1/H},$$

where $H = \sup q_k < \infty$ and $a \in l(q)$. We must show that the map T of Theorem 4.2.2. is bicontinuous. If we write $(a,x) = \sum a_k x_k$, for $a \in l(q)$ and $x \in l(q)$, it enough to prove the following:

(i)$\sup\{|(a,x)| : x \in S(\theta, A)\} < \varepsilon$ whenever $0<h(a)<\min(1,\varepsilon)(A^M+1)^{-1}$, for arbitrary $A > 0$ and $\varepsilon > 0$.

(ii) $[h(a)]^{H(M-1)} \leq A^{-1}$, whenever $\sup\{|(a,x)| : x \in S(\theta,A)\} < 1$, for abritrary $A>1$.

In (i) and (ii), $S(\theta,A)$ denotes the sphere in $l(p)$ with centre the origin and radus$A$. To prove (i) we note that $0<h(a)<1$ implies

$$|a_k x_k| \leq \left(|x_k|^{p_k} + |a_k|^{q_k h - H}\right)h,$$

whence $$|(a,x)| \leq (g^M + 1)h \leq (A^M + 1)h < \varepsilon,$$

for $x \in S(\theta,A)$, whenever $0<h = h(a) < \min(1, \varepsilon)\,(A^m + 1)^{-1}$.

To prove (ii) we take $h(a)>0$ and consider $x^{(n)} \in S(\theta, a)$, where

$$x_k^{(n)} = \frac{(\operatorname{sgn} a_k)|a_k|^{q_k - 1} A^{1/p_k}}{h^{H/p_k}} \quad (1 \leq k \leq n)$$

$$= 0 \qquad (k>n),$$

for $n=1, 2, \ldots$ Then for every $n \geq 1$, whenever $\sup\{|(a,x)| : x \in S(\theta,A)\} < 1$,

$$\sum_{k=1}^{n} |a_k|^{q_k} A^{1/p_k} h^{-H/p_k} < 1.$$

Since $A^{1/M} \leq A^{1/p_k}$, we get

$$\sum_{k=1}^{\infty} |a_k|^{q_k} h^{-H/p_k} \leq A^{-1/M},$$

which clearly implies $h(a) < 1$, whence $h^{1/p_k} \leq h^{1/M}$, *and so*

$$h^{H(1-/M)} \leq A^{-1/M},$$

which is (ii).

By combining Theorem 4.2.3. and the relevant results of Simons [35], who considered the case $0 < p_k \leq 1$, we can give a complete description of $l^*(p)$ in the most general case.

Theorem 4.2.5. If $0<p_k\leq \sup p_k<\infty$ then $l^*(p)$ is linearly homeomorphic to $\hat{M}(p)$.

Theorem 4.2.6. Let $p_k>0$. Then $c_0^{+}(p) = M_0(p)$. When $\sup p_k < \infty$, $c_0^{*}(p)$ is isomphic to $M_0(p)$ and when, in addition, $\inf p_k> 0$, c_0^{*} is linearly homeomorphic to l_1.

Proof. Let $a \in M_0(p)$ and $x \in c_0(p)$. Then $\sum|a_k|N^{-1/p_k} < \infty$, for some N>1 and so $|x_k|^{p_k}<N^{-1}$ *for all sufficiently large k*, whence for such k, $|a_k x_k| \leq |a_k|N^{-1/p_k}$. Consequently, $M_0(p) \subset c_0^{+}(p)$. Now the convergence of $\sum a_k x_k$ for all x in $c_0(p)$ implies that $a \in M_0(p)$. For otherwise, as in the proof of Theorem 4.2.1. we can easily construct a sequence $x \in c_0(p)$ such that $\sum a_k x_k$ diverges. This shows that $c_0^{+}(p) = M_0(p)$.

Now $\sup p_k < \infty$ implies $c_0(p)$ is a topological linear space and it is clear that each f in $c_0^{*}(p)$ is representable as $f(x) = \sum a_k x_k$, for x in $c_0(p)$. By the first part of the theorem it follows that $a \in M_0(p)$. Also by the first part of the theorem we see that

$\sum a_k x_k$ is linear on $c_0(p)$, whenever $a \in M_0(p)$. Now $g(x) < N^{-1/M}$, *where* N > *1, implies* $|x_k| \leq g(x)$ and so

$$\sum |a_k x_k| \leq g(x) \sum_1^{\infty} |a_k| + \sum_{n+1}^{\infty} |a_k| N^{-1/p_k} .$$

When $a \in M_0(p)$ we choose n so large and $g(x)$ so small that $\sum |a_k x_k|$ is as small as we please. It follows that $\sum a_k x_k$ is continuous on $c_0(p)$, whenever $a \in M_0(p)$. Hence the map $f \rightarrow a$ is a linear bijection.

Finally, suppose $p = \inf p_k > 0$. We take the usual norm in l_1. If $g(x) \leq A$ and k is such that $|x_k| < 1$ then $|x_k| \leq A$, and if $|x_k| \geq 1$ then $|x_k|^p \leq |x_k|^{p_k}$, whence

$$|x_k| \leq B = \max (A, A^{M/p}) \qquad \text{for all } K,$$

so that

$$\sum |a_k x_k| \leq B \sum |a_k|.$$

Thus the map $a \rightarrow f$ is continuous.

Now take $A > 1$ and suppose

$$\text{Sup } \{|(a,x)|: x \in S(\theta, A)\} < 1,$$

Where $S(\theta, A)$ is a sphere in $c_0(p)$. Then $(x^{(n)}) \in S(\theta, A)$, where

$$x_k^{(n)} = (\text{sgn} a_k) A^{1/p_k}, \; 1 \leq k \leq n, \qquad x_k^{(n)} = 0, \; k > n.$$

Hence

$$\sum_1^{\infty} |a_k| A^{1/p_k} \leq 1,$$

$$\sum_{1}^{\infty}|a_k| \leq A^{-1/M},$$

since $A > 1$. Consequently, the map $f \to a$ is continuous.

We may also observe that $c_0^{\cdot}(p)$ is normable when $0 < \inf p_k \leq \sup p_k < \infty$, with

$$\|f\| = \sup\{|f(x)| : g(x) \leq 1\} = \sum|a_k|.$$

Hence $c_0^{\cdot}(p)$ is isometrically isomorphic to l_1 in this case.

Theorem 4.2.7. Let $p_k > 0$ for every k. Then $l_\infty^{\dagger}(p) = M_\infty(p)$, where

$$M_\infty(p) = \bigcap_{N=2}^{\infty}\{a : \sum|a_k|N^{1/p_k} < \infty\}.$$

Proof. Let $a \in M_\infty(p)$ *and* $x \in l_\infty(p)$. We choose an integer $N > \max(1, \sup_k |x_k|^{p_k})$. Then $|\sum a_k x_k| \leq \sum|a_k|N^{1/p_k} < \infty$ and therefore, $M_\infty(p) \subset l_\infty^{\dagger}(p)$.

On the other hand if $a \in l_\infty^{\dagger}(p)$ but $a \notin M_\infty(p)$ then there exists an integer $N > 1$ such that $\sum|a_k|N^{1/p_k} = \infty$. Putting $x_k = N^{1/p_k} \operatorname{sgn} a_k$ we have $x \in l_\infty(p)$ but $\sum a_k x_k$ divergent. Hence $l_\infty^{\dagger}(p) = M_\infty(p)$.

CHAPTER V

MEASURES OF NONCOMPACTNESS

5.1. Introduction

In this chapter we shall study the notation of measure of noncompactness (α-measure or set-measure), introduced by Kuratowski [17], and the associated notion of an α-contraction, have proved useful in several areas of functional analysis, operator theory and differential equations. We start with some results from Kuratowski [17, 18].

Usually it is complicated to find the exact value of $\alpha(Q)$. Another measure of noncompactness, which is more applicable in many cases, were introduced and studied by Goldenstein, Gohberg and Markus (the ball measures of noncompatness, Hausdorff measure of noncompactness) [8] in 1957.

Istratescu's measure of noncompactness is closely related to the Hausdorff and Kuratowski measures of noncompactness. Before we give its definition, we need to recall that a bounded subset Q of a complete metric space (X,d) is said to be ε-discrete if $d(x,y) \geq \varepsilon$ for all $x,y \in Q$ with $x \neq y$. Obviously, the set Q is relatively compact if and only if every ε-discrete set is finite for all ε>0.

Now we shall point out the well-know result of Goldenstein, Gohberg and Markus [8, Theorem 1.] concerning the Hasudorff measure of noncompactness in Banach spaces with Schauder basis. Let X be a Banach space with a Schauder basis $\{e_1,e_2,\ldots\}$. Then each element $x\in X$ has a unique representation $x=\sum_{i=1}^{\infty}\phi_i(x)e_i$ where the functions ϕ_i are the basis functionals.

Let $P_n : X\to X$ be the projector onto the linear span of $\{e_1,e_2,\ldots,e_n\}$, that is $P_n(x)=\sum_{i=1}^{n}\phi_i(x)e_i$. Then, in view of the Banach-Steinhaus theorem, all operators P_n and $I\text{-}P_n$ are equibounded.

Finally, we shall measure the noncompactness of an operator.

5.2. The Kuratowski measure of noncompactness

Definition 5.2.1. Let (X,d) be a metric space and Q a bounded subset of X. Then *the Kuratowski measure of noncompactness of Q*, denoted by $\alpha(Q)$, is the infimum of the set of all numbers ε>0 such that Q can be covered by a finite number of sets with diameters < ε, that is

$$\alpha(Q)=\inf\left\{\varepsilon>0:Q\subset\bigcup_{i=1}^{n}S_i, S_i\subset X,\ diam\ (S_i)<\varepsilon (i=1,\dots,n; n\in \mathbb{N})\right\}.$$

The function α is called *Kuratowski's measure of noncompactness.* Clearly

(5.2.1.1) $\alpha(Q)\le$ diam (Q) for each bounded subset Q of X.

As an immediate consequence of Definition 5.2.1, we obtain.

Lemma 5.2.2. Let Q, Q_1 and Q_2 be bounded subsets of a complete metric space (X,d). Then:

(5.2.2.1) $\alpha(Q) = 0$ if and only if $\overline{Q}$ is compact,

(5.2.2.2) $\alpha(Q) = \alpha(\overline{Q})$,

(5.2.2.3) $Q_1\subset Q_2$ implies $\alpha(Q_1)\le\alpha(Q_2)$,

(5.2.2.4) $\alpha(Q_1\cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}$,

(5.2.2.5) $\alpha(Q_1\cap Q_2)\le\min\{\alpha(Q_1), \alpha(Q_2)\}$.

Proof. The statements in (5.2.2.1) and (5.2.2.3) follow from Definition 5.2.1. Clearly $\alpha(Q)\le\alpha(\overline{Q})$. *Let* $\varepsilon>0, S_i$ be a bounded subset of X with diam$(S_i)<\varepsilon$ for $i=1,\dots,n$, and $Q\subset\bigcup_{i-1}^{n}S_i$. Then $\overline{Q}\subset\overline{\bigcup_{i=1}^{n}S_i}=\bigcup_{i=1}^{n}\overline{S}_i$. Since diam$(S_i)$=diam$(\overline{S}_i)$, we conclude $\alpha(\overline{Q})\le\alpha(Q)$. This proves equality (5.2.2.2.).

From (5.2.2.3.) we have $\alpha(Q_1) \le \alpha(Q_1 \cup Q_2)$ and $\alpha(Q_2) \le \alpha(Q_1 \cup Q_2)$: and so

$$\text{(5.2.2.6)} \qquad \max \{\alpha(Q_1), \alpha(Q_2)\} \le \alpha(Q_1 \cup Q_2).$$

Let $\max\{\alpha(Q_1), \alpha(Q_2)\} = s$ and $\varepsilon > 0$. By definition 5.2.1 we know that Q_1 and Q_2 can be covered by a finite number of subsets of diameter smaller than $s+\varepsilon$. Obviously, the union of these covers is a finite cover of $Q_1 \cup Q_2$. Hence, we have $\alpha(Q_1 \cup Q_2) \le s+\varepsilon$, and now we obtain (5.2.2.4) from (5.2.2.6). From $Q_1 \cap Q_2 \subset Q_1$ and $Q_1 \cap Q_2 \subset Q_2$ we obtain $\alpha(Q_1 \cap Q_2) \le \alpha(Q_1)$ and $\alpha(Q_1 \cap Q_2) \le \alpha(Q_2)$. Hence $\alpha(Q_1 \cap Q_2) \le \min\{\alpha(Q_1), \alpha(Q_2)\}$. This proves inequality (5.2.2.5).

The next theorem is a generalization of the well- known Cantor intersection theorem.

Theorem 5.2.3. (Kuratowski [17]) Let (X,d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X such that $\lim_{n\to\infty} \alpha(F_n) = 0$, then the intersection $F_\infty = \bigcap_{n=1}^{\infty} F_n$ is a nonempty and compact subset of X.

Proof. The set F_∞ is a closed subset of X. Since $F_\infty \subset F_n$ for all n = 1,2,..., we obtain from (5.2.2.1) and (5.2.2.3) that F_∞ is a compact set. Now we show $F_\infty \ne \phi$. Let $x_n \in F_n (n = 1,2,...)$ and X_n=

$\{x_i : i \geq n\}$ for $n=1,2\ldots$. Since $X_n \subset F_n$, we obtain from (5.2.2.1), (5.2.2.3) and (5.2.2.4)

(5.2.3.1) $\quad \alpha(X_1) = \alpha(X_n) \leq \alpha(F_n)$ for each n.

The assumption of the theorem and (5.2.3.1) together imply $\alpha(X_1) = 0$, hence X_1 is a relatively compact set. Thus the sequence (x_n) has a convergent subsequence with limit $x \in X$, say. Since F_n is closed in X, we get $x \in F_n$ for all $n = 1,2,\ldots$, that is $x \in F_\infty$.

If X is a normed space, then the function α has some additional properties connected with the vector (linear) structures of a normed space [5].

Theorem 5.2.4. (Darbo [5]) Let Q, Q_1 and Q_2 be bounded subsets of a normed space X. Then:

(5.2.4.1) $\quad \alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2)$,

(5.2.4.2) $\quad \alpha(Q + x) = \alpha(Q)$ for each $x \in X$,

(5.2.4.3) $\quad \alpha(\lambda Q) = |\lambda| \alpha(Q)$ for each $\lambda \in \mathbb{F}$,

(5.2.4.4) $\quad \alpha(Q) = \alpha(co(Q))$.

Proof. Let S_i be a bounded subset of X with diam$(S_i) < d$ for each $i=1,\ldots,n$ and $Q_1 \subset \bigcup_{i=1}^{n} S_i$. Furthermore, let G_j be a bounded subset of X with diam$(G_j) < p$ for each $j=1,\ldots,m$ and $Q_2 \subset \bigcup_{j=1}^{m} G_j$. Then

$$(5.2.4.5) \qquad Q_1 + Q_2 \subset \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(S_i + G_j) \text{ and diam } (S_i + G_j) < d + p.$$

It follows from (5.2.4.6) that $\alpha(Q_1 + Q_2) < d + p$. This shows inequality (5.2.4.1). Let $x \in X$. By (5.2.4.1) it follows that

$$(5.2.4.6) \qquad \alpha(Q + x) \leq \alpha(Q) + \alpha(\{x\}) = \alpha(Q),$$

and by the same argument we have

$$(5.2.4.7) \qquad \alpha(Q) = \alpha((Q + x) + (-x)) \leq \alpha(Q + x) + \alpha(\{-x\}) = \alpha(Q + x).$$

Now we obtain (5.2.4.2) from (5.2.4.6) and (5.2.4.7).

For $\lambda = 0$, equality (5.2.4.3) is obvious. Let S_i be a bounded subset of X with diam(S_i) < d for i=1,...,n and $Q_1 \subset \bigcup_{i=1}^{n} S_i$. Then for any $\lambda \in I\!F$, $\lambda Q \subset \bigcup_{i=1}^{n} \lambda S_i$ and $\text{diam}(\lambda S_i) = |\lambda| \text{ diam } S_i$. Hence it follows that $\alpha(\lambda Q) \leq |\lambda| \alpha(Q)$. If $\lambda \neq 0$, analogously we have $\alpha(Q) = \alpha(\lambda^{-1}(\lambda Q)) \leq |\lambda^{-1}| \alpha(\lambda Q)$, that is $|\lambda| \alpha(Q) \leq \alpha(\lambda Q)$. This proves (5.2.4.3).

Now we prove (5.2.4.4). Clearly $\alpha(Q) \leq \alpha(co(Q))$, and it suffices to show $\alpha(co(Q) \leq \alpha(Q)$. Let S_i be a bounded subset of X with diam(S_i) < d for each i = 1,...,n and $Q = \bigcup_{i=1}^{n} S_i$. By Theorem 1.4.2 it follows that

$$(5.2.4.8) \qquad co(Q) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, x_i \in co(S_i) \ (i = 1,\ldots,n \right\}.$$

Let ε>0 and $S=\{(\lambda_1,\ldots,\lambda_n):\sum_{i=1}^{n}\lambda_i=1,\lambda_i\geq 0(i=1,\ldots,n)\}$. Then S is a compact subset of $(R^n,\|\cdot\|_\infty)$, where $\|(\lambda_1,\ldots,\lambda_n)\|_\infty=\sup_{1\leq i\leq n}|\lambda_i|$. We put $M=\sup\{\|x\|:x\in\bigcup_{i=1}^{n}co(S_i)\}$. Let T = $\{(t_{j,1},\ldots,t_{j,n}):j=1,\ldots,m\}\subset$S be a finite $\varepsilon/(Mn)$-net for S, with respect to the $\|\cdot\|_\infty$- norm. Hence, if $\sum_{i=1}^{n}\lambda_i x_i$ is a convex combination of elements of Q, where we suppose that $x_i\in co(S_i)$ for i = 1,...,n, then there exists $(t_{j,1},\ldots,t_{j,n})\in$T such that

(5.2.4.9) $$\|(\lambda_1,\ldots,\lambda_n)-(t_{j,1},\ldots,t_{j,n})\|_\infty<\frac{\in}{M}n.$$

Since

(5.2.4.10) $$\sum_{i=1}^{n}\lambda_i x_i=\sum_{i=1}^{n}t_{j,i}x_i+\sum_{i=1}^{n}(\lambda_i-t_{j,i})x_i,$$

it follows from (5.2.4.7), (5.2.4.8) and (5.2.4.9) that

(5.2.4.11) $$co(Q)\subset\bigcup_{j=1}^{m}\left\{\sum_{i=1}^{n}t_{j,i}co(S_i)\right\}+\frac{\varepsilon}{Mn}\sum_{i=1}^{n}B_i,$$

where $B_i=\{x\in X:\|x\|\leq M\}$ for $i=1,2,\ldots,n$. Now, by (1.4.2.3), (1.4.2.4), (5.2.4.1), (5.2.4.3), Corollary 1.4.4 and (5.2.5.1) we have

$$\alpha(co(Q))\leq\alpha\left(\bigcup_{j=1}^{m}\left\{\sum_{i=1}^{n}t_{j,i}co(S_i)\right\}\right)+\alpha\left(\frac{\varepsilon}{Mn}\sum_{i=1}^{n}B_i\right)$$

$$\leq \max_{1\leq j\leq m} \alpha\left(\sum_{i=1}^{n} t_{j,i} co(S_i)\right) + \frac{\varepsilon}{Mn}\sum_{i=1}^{n}\alpha(B_i)$$

$$< \max_{1\leq j\leq m} \sum_{i=1}^{n} t_{j,i}\alpha(co(S_i)) + \frac{\varepsilon}{Mn} 2nM$$

$$< d \max_{1\leq j\leq m} \sum_{i=1}^{n} t_{j,i} + 2\varepsilon < d + 2\varepsilon .$$

Theorem 5.2.5. (Furi-Vignoli [7], Nussbaum [31]) Let X be an infinite-dimensional normed space. Then $\alpha(B_X) = 2$.

Proof. Clearly $\alpha(B_X) \leq 2$. If $\alpha(B_X) < 2$, then there exist bounded and closed subsets Q_i of X with diam (Q_i) <2 for $i=1,\dots,n$ such that $B_X \subset \bigcup_{i=1}^{n} Q_i$. Let $\{x_1,\dots x_n\}$ be a linearly independent subset of X and E_n be the set of all linear combinations of elements of the set $\{x_1,\dots,x_n\}$ with real coefficients. Clearly E_n is a real n-dimensional normed space (the norm on E_n, of course, being the restriction of the norm on X). By $S_n = \{x \in E_n : \| x \| = 1\}$, we denote the unit sphere of E_n. Let us mention that $S_n \subset \bigcup_{i=1}^{n} S_n \cap Q_i$, $diam(S_n \cap Q_i) < 2$ and $S_n \cap Q_i$ is a closed subset of E_n for each $i=1,\dots n$. This is a contradiction to the well-known Ljusternik-Šnirelman-Borsuk theorem (see the proof in [6], pp. 303-307]: If S_n is the unit sphere of an n-dimensional real normed space E_n, F_i a closed subset of E_n for each $i=1,\dots,n$ and $S_n \subset \bigcup_{i=1}^{n} F_i$, then there exists $i_0 \in \{1,\dots,n\}$ such that the set $S_n \cap F_{i_0}$ contains a pair of antipodial points, that is, there exists $x_0 \in S_n \cap F_{i_0}$, such that $\{x_0, -x_0\} \subset S_n \cap F_{i_0}$.

5.3. The Hausdorff measure of noncompactness

Definition 5.3.1. Let (X,d) be a metric space and Q a bounded subset of X. Then *the Hausdorff measure of noncompactness of the set Q*, denoted by $\chi(Q)$ is defined to be the infimum of the set of all reals ε>0 such that Q can be covered by a finite number of balls of radii<ε, that is

(5.3.1.1.) $\chi(Q) = \inf\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in X, r_i < \in (i = 1,\ldots n) n \in I\!N\}$.

The function χ is called *Hausdorff measure of noncompactness*.

Let us remark that in the definition of the Hausdorff measure of noncompactness of the set Q it is not supposed that centres of the balls which cover Q belong to Q. Hence, (5.3.1.1) can equivalently be stated as follows.

(5.3.1.2) $\chi(Q) = \inf\{\varepsilon > 0 : Q \textit{ has a finite } \varepsilon\textit{-net in } X\}$.

The Hausdorff measure of noncompactness is often called *ball measure of noncompactness*. The next lemma and theorem could be proved analogously as in the case of the Kuratowski measure of noncompactness.

Lemma 5.3.2. Let Q, Q_1 and Q_2 be bounded subsets of the metric space (X,d). Then

(5.3.2.1) $\chi(Q)=0$ if and only if Q is totally bounded,

(5.3.2.2) $\chi(Q)=\chi(\overline{Q})$,

(5.3.2.3) $Q_1 \subset Q_2$ implies $\chi(Q_1) \leq \chi(Q_2)$,

(5.3.2.4) $\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$,

(5.3.2.5) $\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}$.

Theorem 5.3.3. Let Q, Q_1 and Q_2 be bounded subsets of the normed space X. Then

(5.3.3.1) $\chi(Q_1+Q_2) \leq \chi(Q_1) + (Q_2)$,

(5.3.3.2) $\chi(Q+x) = \chi(Q)$ for each $x \in X$,

(5.3.3.3) $\chi(\lambda Q) = |\lambda|\chi(Q)$ for each $\lambda \in \mathbb{F}$,

(5.3.3.4) $\chi(Q) = \chi(co(Q))$.

The next theorem shows that the functions α and χ are in some sense equivalent.

Theorem 5.3.4. Let (X,d) be a metric space and Q be a bounded subset of X. Then

(5.3.4.1) $\chi(Q) \leq \alpha(Q) \leq 2\chi(Q)$.

Proof. Let $\varepsilon > 0$. If $\{x_1,\ldots,x_n\}$ is an ε-net of Q, then $\{Q \cap B(x_i,\in)\}_{i=1}^{n}$ is a cover of Q with sets of diameter $<2\varepsilon$. This shows $\alpha(Q) \leq 2\chi(Q)$. To prove the left side inequality in (5.3.4.1), let us suppose that $\{S_i\}_{i=1}^{k}$ is a cover of Q with sets of diameter $< \varepsilon$ and $y_i \in S_i$ for i=1,...k. Now $\{y_1,\ldots,y_k\}$is an $\varepsilon-net$ of Q. This proves $\chi(Q) \leq \alpha(Q)$.

Remark 5.3.5. The inequalities (5.3.4.1) are best possible in general, as an example shows. These measures are closely related to geometric properties of the space and it is possible to improve the inequality $\chi(Q) \leq \alpha(Q)$ in certain spaces. For example in Hilbert space, $\sqrt{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q)$, and in l_p for $1 \leq p < \infty$, $\sqrt[p]{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q)$.

Theorem 5.3.6. Let X be an infinite-dimensional normed space and B_X be the closed unit ball of X. Then $\chi(B_X)$=1.

Proof. Obviously $\chi(B_X) \leq 1$. If $\chi(B_X) = q < 1$, then we choose ε>0 such that $q+\varepsilon$<1. Now there exists a $(q+\varepsilon)$-net of B_X, say $\{x_1,\ldots,x_k\}$. Hence

(5.3.6.1) $$B_X \subset \bigcup_{i=1}^{k}\{x_i + (q+\varepsilon)B_X\}.$$

Now it follows from Lemma 5.3.2 and Theorem 5.3.3 that

(5.3.6.2) $$q = \chi(B_X) \leq \max_{1\leq i\leq k} \chi(\{x_i + (q+\varepsilon)B_X\}) = (q+\varepsilon)q.$$

Since $q+\varepsilon<1$, by (5.3.6.1) we have $q=0$, that is B_X is a totally bounded set. But this is impossible since X is an infinite-dimensional space. Hence $\chi(B_X)=1$.

Now we shall show how to compute the Hausdorff measure of noncompactness in the spaces l_p $for\, 1\le p<\infty\, and\, c_0$.

Theorem 5.3.7. Let Q be a bounded subset of the normed space X, where X is l_p for $1\le p<\infty$ or c_0. If $P_n : X\to X$ is the operator defined by $P_n(x_1,x_2,\ldots)=(x_1,x_2,\ldots,x_n,0,0,\ldots)$ for $(x_1,x_2,\ldots)\in X$; then

(5.3.7.1) $$\chi(Q)=\lim_{n\to\infty}\sup_{x\in Q}\|(I-P_n)x\|.$$

Proof. Clearly

(5.3.7.2) $$Q\subset P_nQ+(I-P_n)Q.$$

It follows from Lemma 5.3.2, Theorem 5.3.3 and (5.3.7.2) that

(5.3.7.3) $$\chi(Q)\le\chi(P_nQ)+\chi((I-P_n)Q)=\chi((I-P_n)Q)\le\sup_{x\in Q}\|(I-P_n)x\|.$$

Since the limit in (5.3.7.1) clearly exists, we have by (5.3.7.3)

(5.3.7.4) $$\chi(Q)\le\lim_{n\to\infty}\sup_{x\in Q}\|(I-P_n)x\|.$$

Now we prove the converse inequality in (5.3.7.4). Let $\varepsilon>0$ and $\{z_1,\ldots,z_k\}$ be a $[\chi(Q)+\varepsilon]-net$ of Q. It is easy to prove that

(5.3.7.5) $$Q\subset\{z_1,\ldots,z_k\}+[\chi(Q)+\varepsilon]B_X.$$

It follows from (5.3.7.5) that for any $x \in Q$ there exist $z \in \{z_1,\ldots,z_k)$ and $s \in B_X$ such that $x = z + [\chi(Q)+\varepsilon]s$. Hence

$$\sup_{x\in Q} \| (I-P_n)x \| \le \sup_{1\le i\le k} \| (I-P_n)z_i \| + [\chi(Q)+\varepsilon]. \tag{5.3.7.6}$$

Finally, (5.3.7.6) implies $\lim_{n\to\infty} \sup_{x\in Q} \| (I-P_n)x \| \le \chi(Q)+\varepsilon$.

Theorem 5.3.8. Let l_∞ be the real normed space of bounded sequences with sub-norm and Q be a bounded subset of l_∞. Then $\alpha(Q) = 2\chi(Q)$.

Proof. We know that $\alpha(Q) \le 2\chi(Q)$. Let $\varepsilon > 0$ and $Q_1,\ldots,Q_n$ be subsets of $l_\infty(\mathbb{R})$ such that $Q \subset \bigcup_{i=1}^{n} Q_i$ and diam $Q_i < \alpha Q) + \varepsilon$. For any $k \in \mathbb{N}$ we put $\alpha_{k,i} = \inf\{x_k : (x_j) \in Q_i\}$, $\beta_{k,i} = \sup\{x_k : (x_j) \in Q_i\}$, $c_{k,i} = (\alpha_{k,i} + \beta_{k,i})/2$, $B_i = B((c_{k,i})_{k=1}^{\infty}),(\alpha(Q)+\varepsilon)/2)$ for i=1,…,n. It is easy to prove that $Q_i \subset B_i$. Hence $\chi(Q) \le (\alpha(Q)+\varepsilon)/2$, that is $2\chi(Q) \le \alpha(Q)$.

We shall prove that the Hausdorff measure of noncompactness is connected with the Hausdorff distance.

Theorem 5.3.9. Let (X,d) be a metric space. Then (M_X^c, d_H) is a metric space.

Proof. Clearly d_H (S,Q) = 0 if and only if S = Q, and $d_H(S,Q)$ = d_H (Q,S) for all $S,Q \in \mathsf{M}_X^c$.

To show the triangle inequality, suppose $S,Q,F \in \mathsf{M}_X^c, x \in S, y \in Q$ and $z \in F$. It is easy to prove $d(x,F) \le d(x,y) + d(y,F) \le d(x,y) + d_H(Q,F)$, and this implies

$$d(x,F) \le \inf_{y \in Q} d(x,y) + d_H(Q,F) = d(x,Q) + d_H(Q,F)$$

(5.3.9.1) $\qquad \le d_H(S,Q) + d_H(Q,F).$

Replacing x and F by z and S in (1.6.9.1), respectively, we obtain

(5.3.9) $\qquad d(z,S) \le d_H(F,Q) + d_H(Q,S)$

Finally, (5.3.9.1) and (5.3.9.2) together imply $d_H\ (S,F)$ $\le d_H(S,Q) + d_H(Q,F)$.

Theorem 5.3.10. Let (X,d) be a metric space, $Q,Q_1,Q_2 \in \mathsf{M}_X$, and N_X^c be the set of all nonempty and compact subsets of (X,d) Then

(5.3.10.1) $\qquad |\chi(Q_1) - \chi(Q_2)| \le d_H(Q_1,Q_2)$,

(5.3.10.2) $\qquad \chi(Q) = d_H(Q, N_X^c)$.

Proof. Let $\varepsilon > 0$ and $d = d_H(Q_1,Q_2)$. Then it follows from (5.3.1.2) and (1.4.1) that there exists a finite set $S \subset X$, such that

(5.3.10.3) $\qquad Q_1 \subset B(Q_2, d+\varepsilon)$ and $Q_2 \subset B(S, \chi(Q_2)+\varepsilon)$.

Furthermore, (5.3.10.3) implies

(5.3.10.4) $\qquad Q_1 \subset B(S, d+\chi(Q_2)+2\varepsilon)$, and so we conclude

(5.3.10.5) $$\chi(Q_1) \leq \chi(Q_2) + d + 2\varepsilon.$$

Now (5.3.10.1) clearly follows from (5.3.10.5).

To prove (5.3.10.2), let us remark that the inequality $\leq$ in (5.3.10.2) follows from (5.3.10.1). Therefore it suffices to show the inequality $\geq$. If $\varepsilon>0$ then there exists a finite set $F \subset X$, such that

(5.3.10.6) $$Q \subset B(F, \chi(Q)+\varepsilon) \text{ and } F \subset B(Q, \chi(Q)+\varepsilon).$$

Now (5.3.10.6) and theorem 1.4.2 together imply $d_H(Q, N_X^c) \leq d_H(Q, F) \leq \chi(Q)+\varepsilon$.

Corollary 5.3.11. Let N_X^c be the set of all nonempty and compact subsets of a complete metric space (X,d). Then N_X^c is a closed subset of (M_X^c, d_H).

Definition 5.3.12. Let (X,d) be a metric space and Q a bounded subset of X. Then *the inner Hausdorff measure of noncompactness of the set Q*, denoted by $\chi_i(Q)$ is defined to be the infimum of the set of all reals $\varepsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \varepsilon$ and centers in Q, that is

$$\chi_i(Q)=\inf\left\{\varepsilon>0:Q\subset\bigcup_{i=1}^{n}B(x_i,r_i),x_i\in Q,r_i<\varepsilon(i=1,\dots,n)n\in N)\right\}.$$

The function χ_i is called <u>*inner Hausdorff measure of noncompactness.*</u> Hence the formula in Definition 5.3.1.2 can equivalently be stated as follows.

$$\chi_i(Q)=\inf\{\varepsilon>0:Q \text{ has a finite } \varepsilon\text{-net in } Q\}.$$

If Q,Q_1 and Q_2 are bounded subsets of the metric space (X,d), then

$$\chi_i(Q)=0 \text{ if and only if } Q \text{ is totally bounded,}$$

$$\chi_i(Q)=\chi_i(\overline{Q}),$$

but in general

$$Q_1\subset Q_2 \text{ does not imply } \chi_i(Q_1)\le\chi_i(Q_2).$$

and

$$\chi_i(Q_1\cup Q_2)\ne\max\{\chi_i(Q_1),\chi_i(Q_2)\}.$$

Let Q,Q_1 and Q_2 be bounded subset of the normed space X. Then

$$\chi_i(Q_1+Q_2)\le\chi_i(Q_1)+\chi_i(Q_2),$$

$$\chi_i(Q+x)=\chi_i(Q) \text{ for each } x\in X,$$

$$\chi_i(\lambda Q)=|\lambda|\chi_i(Q) \text{ for each } \lambda\in\mathbb{F},$$

but in general

$$\chi_i(Q) \neq \chi_i(co(Q)).$$

Definition 5.3.13. (Istrăţescu, [10]) Let (X,d) be a complete metric space and Q a bounded subset of X. Then the Istrăţescu measure of noncompactness (β-measure, I-measure) of Q, is denoted by β(Q), and defined by

$$\beta(Q) = \inf\{\varepsilon > 0 : Q \text{ has no infinite } \varepsilon\text{-discrete subsets}\}.$$

The function β is called _Istrăţescu's measure of noncompactness._ Let us remark [4] that β can be defined also by

$$\beta(Q) = \sup\{\varepsilon > 0 : Q \text{ contains an infinite } \varepsilon\text{-discrete set}\},$$

and the above mentioned properties of α are also valid for β (see e.g.[4]).

Theorem 5.3.14. (Daneš,[4]) Let (X,d) be a metric space and Q be a bounded subset of X. Then

$$\chi(Q) \leq \chi_i(Q) \leq \beta(Q) \leq \alpha(Q) \leq 2\chi(Q).$$

Hence, in particular $\frac{1}{2}\alpha(Q) \leq \beta(Q) \leq \alpha(Q)$ and $\chi(Q) \leq \beta(Q) \leq 2\chi(Q)$.

Theorem 5.3.15. (Goldenštein, Gohberg and Markus [8]) Let X be a Banach space with a Schauder basis $\{e_1,e_2,\ldots\}$, Q be a bounded subset of X, and $P_n : X \to X$ the projector onto the linear span of $\{e_1,e_2,\ldots,e_n\}$.Then

$$\text{(5.3.15.1)} \qquad \frac{1}{a}\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right)\le\chi(Q)\le$$

$$\le\inf_n\sup_{x\in Q}\|(I-P_n)(x)\|\le\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right),$$

where $a=\limsup_{n\to\infty}\|I-P_n\|$.

Proof. Clearly, for any natural number n we have

$$\text{(5.3.15.2)} \qquad Q\subset P_nQ+(I-P_n)Q.$$

It follows from Lemma 5.3.2 and Theorem 5.3.3 and (5.3.15.2) that

$$\text{(5.3.15.3)}\ \chi(Q)\le\chi(P_nQ)+\chi((I-P_n)Q)=\chi((I-P_n)Q)\le\sup_{x\in Q}\|(I-P_n)(x)\|.$$

Now we obtain

$$\text{(5.3.15.4)} \quad \chi(Q)\le\inf_n\sup_{x\in Q}\|(I-P_n)(x)\|\le\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right),$$

Hence it suffices to show the first inequality in (5.3.15.1). Let $\varepsilon>0$ and $\{z_1,\dots,z_k\}$ be a $[\chi(Q)+\varepsilon]-$*net* of Q. It is easy to show that $Q\subset\{z_1,\dots,z_k\}+[\chi(Q)+\varepsilon]\ B_X$. This implies that for any $x\in Q$ there exist $z\in\{z_1,\dots,z_k\}$ and $s\in B_X$ such that $x=z+[\chi(Q)+\varepsilon]s$, and so

$$\sup_{x\in Q}\|(I-P_n)(x)\|\le\sup_{1\le i\le k}\|(I-P_n)(z_i)\|+[\chi(Q)+\varepsilon]\,\|(I-P_n)\|.$$

This implies

$$\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right)\leq(\chi(Q)+\varepsilon)\limsup_{n\to\infty}\|I-P_n\|.$$

5.4. Operators. So far we "measured" the noncompactness of a bounded subset of a metric space. Now we "measure" the noncompactness of an operator.

Definition. 5.4.1. Let κ_1 and κ_2 any of the measures of noncompactness defined above on the Banach spaces X and Y, respectively. An operator $L: X \to Y$ is said to be *(κ_1,κ_2)-bounded* if

(5.4.1.1) $\quad L(Q) \in \mathsf{M}_Y$ for each $Q \in \mathsf{M}_X$

and there exists a real k with $0 \leq k < \infty$ such that

(5.4.1.2) $\quad \kappa_2(L(Q)) \leq k\kappa_1(Q)$ for each $Q \in \mathsf{M}_X$.

If an opperator L is (κ_1,κ_2)-bounded then the number $\|K\|_{\kappa_1,\kappa_2}$ defined by

(5.4.1.3) $\quad \|L\|_{\kappa_1,\kappa_2} = \inf\{k \geq 0 : \kappa_2(L(Q)) \leq k\kappa_1(Q)$ for each $Q \in \mathsf{M}_X\}$

is called *(κ_1,κ_2)-operator norm of L, or (κ_1,κ_2)-measure of noncompactness* of L, or simply *measures of noncompactness* of L.

If $\kappa_1 = \kappa_2 = \kappa$, then we write $\|L\|_\kappa$ instead of $\|L\|_{\kappa,\kappa}$.

The next theorem is related to the Hausdorff measure of noncompactness.

Theorem 5.4.2. Let X and Y be Banach spaces and $L \in B(X,Y)$. Then $\| L \|_{\chi} = \chi(L(S_X)) = \chi(L(B_X))$.

Proof We write B = B_X and S = S_X. Since co(S) =B_X and L(co(S)) = co(L(S)), it follows from (5.3.3.4) that

(5.4.2.1) $\quad \chi(L(B)) = \chi(L(co(S)) = \chi(co(L(S)) = \chi(L(S)),$

hence we have by (5.4.1.2) and Theorem 3.3.6 $\chi(L(B)) \leq \| L \|_{\chi}$. Now we show $\| L \|_{\chi} \leq \chi(L(B))$. Let $Q \in \mathrm{M}$ and $\{x_1\}_{i=1}^{n}$ be a finite r-net of Q. Then $Q \subset \bigcup_{i=1}^{n} B(x_i, r)$ and obviously

(5.4.2.2) $$L(Q) \subset \bigcup_{i=1}^{n} L(B(x_i, r)).$$

It follows from (5.4.2.2), Lemma 5.3.2 and Theorem 5.3.3 that

$$\chi(L(Q)) \leq \chi\left(\bigcup_{i=1}^{n} L(B(x_i, r)) = \chi(L(B(0, r)) \right) = r\chi(L(B)),$$

and we have $\chi(L(Q)) \leq \chi(Q)\chi(L(B))$.

Corollary 5.4.3. Let X, Y and Z be Banach spaces, $L \in B(X,Y)$, $\widetilde{L} \in B(Y,Z)$ and $\| \cdot \|_K$ the quotient norm on the Banach space $B(X,Y)/K(X,Y)$. Then $\| \cdot \|_{\chi}$ is a seminorm on B(X,Y) and

(5.4.3.1) $\quad \| L \|_{\chi} = 0$ if and only if $L \in K(X,Y)$,

(5.4.3.2) $\quad \| L \|_{\chi} \leq \| L \|,$

(5.4.3.3) $\|L+K\|_\chi=\|L\|_\chi$, for each $K\in K(X,Y)$,

(5.4.3.4) $\|\widetilde{L}\circ L\|_\chi\leq\|\widetilde{L}\|_\chi\|L\|_\chi$.

(5.4.3.5) $\|L\|_\chi\leq\|L\|_K$.

The following results will give a technique for the evaluation of the Hausdorff measure of noncompactness of an operator on the space l_1.

Theorem 5.4.4. We have $L\in B(l_1,l_1)$ if and only if there exists an infinite matrix $A=(a_{nk})_{n,k}^{\infty}$ of complex numbers such that

(5.4.4.1) $\|A\|=\sup_k\sum_{n=0}^{\infty}|a_{nk}|<\infty$

(5.4.4.2) $L(x)=A(x)$ for all $x\in l_1$.

In this case

(5.4.4.3) $\|L\|=\|A\|$,

and the operator L uniquely determines the matrix $A=(a_{nk})_{n,k}^{\infty}$. The operator L is said to be given (defined) by the matrix A.

Proof. First we assume $L\in B(X,Y)$. We write $L_n=P_n\circ L$ for all n where P_n denotes the n-th coordinate, and put $a_{n,k}=L_n(e^{(k)})$ for all n,k = 0,1,.... Since l_1 is a BK space, we have $L_n\in l_1^*$ for each n and so $L_n(x)=A_n(x)$ for each n by Theorem 3.4.2. This yields the representation in ((5.4.4.2). If we choose $x=e^{(k)}$, then

$$\| L(e^{(k)})\|_1=\sum_{n=0}^{\infty}| L_n(e^{(k)})|=\sum_{n=0}^{\infty}| a_{n,k}|\leq\| L\|\|e^{(k)}\|_1=\| L\| \quad \text{for all,}$$

that is

$$(5.4.4.4) \qquad \| A\|=\sup_k\sum_{n=0}^{\infty}| a_{nk}|\leq\| L\|<\infty$$

and (5.4.4.1) holds. Further

$$(5.4.4.5) \qquad \| L(x)\|_1=\sum_{k=0}^{\infty}| A_n(x)|\leq\sum_{k=0}^{\infty}| x_k|\sum_{n=0}^{\infty}| a_{nk}|\leq\| A\|\,\| x\|_1 \text{ for all } x\in l_1$$

and so $\| L\|\leq\| A\|$. This and (5.4.4.4) together yield (5.4.4.3).

Conversely, let condition (5.4.4.1) hold. Then obviously $\sup_n | a_{nk}|<\infty$ for all n, that is $A_n\in X^\beta$ for all n. Let $x\in l_1$. As in (5.4.4.5), we obtain $A(x)\in l_1$ by (5.4.4.1), whence $A\in(l_1,l_1)$. We define the linear operator $L:l_1\to l_1$ by (5.4.4.2). Then $L\in B(l_1,l_1)$ by Theorem 2.3.1(a).

Theorem 5.4.5. (Goldenštein, Gohberg and Markus [8]) Let $L\in B(l_1,l_1)$ be given by an infinite matrix $A=(a_{nk})_{n,k=0}^{\infty}$. Then

$$(5.4.5.1) \qquad \| L\|_\chi=\lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}| a_{nk}|.$$

Proof. We write $S=S_{l_1}$. It follows from Theorems 5.3.7 and 5.4.4 that

(5.4.5.2) $$\|L\|_\chi=\chi(L(S))=\lim_{m\to\infty}\sup_{x\in S}\sum_{n=m}^{\infty}\left|\sum_{k=0}^{\infty}a_{nk}x_k\right|.$$

The limit in (5.4.5.1) obviously exists. From

$$\sup_{x\in S}\sum_{n=m}^{\infty}\left|\sum_{k=0}^{\infty}a_{nk}x_k\right|\le\sup_{x\in S}\sum_{n=m}^{\infty}\sum_{k=0}^{\infty}|a_{nk}x_k|=\sup_{x\in S}\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}|a_{nk}||x_k|$$

$$\le\sup_k\sum_{n=m}^{\infty}|a_{nk}|$$

and (5.4.5.2) we obtain

(5.4.5.3) $$\|L\|_\chi\le\lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}|.$$

To prove the converse inequality, we choose $x=e^{(k)}\in l_1$. Since $L(e^{(k)})=A^k=(a_{nk})_{n=0}^{\infty}$, Theorem 5.3.7 implies

$$\chi(\{L(e^{(k)}):k=0,1,\dots\})=\lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}|\le\chi(L(S)).$$

This and inequality (5.4.5.3) together yield (5.4.5.1).

Corollary 5.4.6. Let $L\in B(l_1,l_1)$ be given by the infinite matrix $A=(a_{nk})_{n,k=0}^{\infty}$. Then L is compact if and only if

$$\lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}|=0.$$

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